CM-2018 Lessons in Classical Mechanics

Cyclic Coordinates, Integration of EOM by Quadratures

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1 What is a cyclic coordinate?

A generalised coordinate q is called cyclic if the Lagrangian L is independent of q. In such a case the equation of motion for q

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\Big) - \frac{\partial L}{\partial q} = 0. \tag{1}$$

Since q is cyclic, we have

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\Big) = 0 \tag{2}$$

In other words the momentum canonically conjugate to q is a constant of motion.

2 Example for Using Conservation laws

The momentum conjugate to cyclic coordinates is a constant of motion. This is very helpful in integrating the equations of motion. Consider the example discussed above in previous Lesson-A03. There we had a particle moving in two dimensions in a potential that depends only on the radial distance from the origin. Changing the variables to plane polar coordinates (r, θ) we had obtained

$$\begin{aligned} \text{K.E.} &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 \\ &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 \qquad (3) \\ \therefore \qquad L &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - V(r) \qquad (4) \end{aligned}$$

2.1 Find all cyclic coordinates and conservation laws

From the expression for Lagrangian in plane polar coordinates it is obvious that θ is a cyclic coordinate. Hence the corresponding momentum p_θ is conserved. Thus

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = \text{constant, say } \ell. \qquad (5)$$

Since the total energy is also a constant of motion, we have

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r) \qquad (6)$$

2.2 Solve for generalized velocities

Solving for $\dot{\theta}$ in terms for ℓ from Eq.(5) we get

$$\dot{\theta} = \frac{\ell}{mr^2} \qquad (7)$$

and substituting in (6) we get

$$E = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} + V(r) \qquad (8)$$

This is a relation between $r, \dot{r}$ and can be solved for $\dot{r}$. This gives

$$\frac{dr}{dt} = \sqrt{\frac{2}{m}\left(E - \frac{\ell^2}{2mr^2} - V(r)\right)} \qquad (9)$$

2.3 Integrate the velocities

Integrating we get a relation between r and t.

$$t = \int \frac{dr}{\sqrt{\frac{2}{m}\Big(E - \frac{\ell^2}{2mr^2} - V(r)\Big)}} + \text{const.} \tag{10}$$

This relation can be inverted to give r as a function of time, $r = r(t)$. Substituting in (7) and integrating gives

$$\theta = \int \frac{\ell}{mr^2(t)}\, dt + \text{const.} \tag{11}$$

On integration this equation can be used to get θ as a function of t. Thus we see that the conservation laws help in integrating the equations of motion. Sufficiently many conservation laws can reduce the solution to quadratures.

3 Eliminating Cyclic Degrees of Freedom

3.1 Routhian

The Routhian is a kind of partial Legendre transform of Lagrangian. In the previous section a Legendre transform was used to pass Lagrangian to a Hamiltonian to description. In this process all the generalised velocities were eliminated in favour of canonical momenta. It possible to do this only for some of the generalized velocities. Let the set of generalised coordinates be denoted as q, ξ and Lagrangian be a function of $q, \xi, \dot{q}, \dot{\xi}$. Define the canonical momenta as usual by

$$p_k = \frac{\partial L}{\partial \dot{q}_k} \tag{12}$$

We introduce Routhian by doing a Legendre transform from $\dot{q}_k$ to p_k.The Routhian is defined by

$$R(q, p, \xi, \dot{\xi}) = \sum_k p_k \dot{q}_k - L \tag{13}$$

In an alternate form of dynamics, the canonical momenta take over the role played by velocities and Hamiltonian becomes central quantity which governs the dynamics.The EOM can be written in an alternate form called the Hamiltonian EOM. In the Hamiltonian dynamics the velocities are eliminated in favour of canonical momenta. Thus the canonical momenta, in this example, coincide with components of momentum $m\dot{\vec{r}}$ and Hamiltonian is equal to the energy. However, it must be remarked that *the canonical momenta are not always equal to 'ordinary' momenta and Hamiltonian need not be a sum of* $K.E + P.E.$

This is the case when the system is described, for example, by a velocity dependent generalised potential. Motion of a charged particle in external magnetic field constitutes an example of this type where the canonical momentum is not equal to ordinary momentum.

3.2 Using Routhian

Let us note that a cyclic coordinate is absent and that the canonically conjugate momentum is a constant, so it should be possible to eliminate these degrees of freedom completely. That this is indeed possible can be seen by doing a 'partial Legendre transform' of the Lagrangian we arrive at a description in terms of canonical momentum which is a constant of motion.

The process described above leads to using Routhian having just the remaining generalised coordinates and velocities and constants. We leave the details to be worked out as an exercise to be worked out by you separately.

4 Points to Remember

- ✍ Solving conservation law equations for velocities and integrating facilitates reducing EOM having second order time derivatives to equations with first order time derivative.
- ✍ If the number of conservation laws equals the number of degrees of freedom, the problem of obtaining solution reduces to quadratures.
- ✍ We have seen, in the example discussed, that use of (7) in the Lagrangian (4), to eliminate $\dot{\theta}$ leads to an expression, which when regarded as Lagrangian for r alone, does not correctly describe the equation of motion.
  One should use Routhian to fully eliminate the cyclic degrees of freedom.