CM-18 Lessons in Classical Mechanics

I.3 Conservation of Energy

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§1 Lets us discuss

1. Give an example of application of energy conservation.

2. What do you understand by conservation law? When we say something $X(q,p)$ is a constant of motion what exactly it means mathematically?

3. Give examples of a system for which energy is not conserved.

4. Consider an example of a particle moving in one dimension in a potential $V(x)$. Taking total time derivative of total energy, $E = \frac{1}{2}m\dot{x}^2 + V(x)$ we would get

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + \frac{dV(x)}{dx}\frac{dx}{dt}. \tag{1}$$

The right hand side does not seem to become zero!
What is missing?
How do we see that the right hand side of (1) is zero?

§2 Energy conservation

If the Lagrangian does not depend on time explicitly,there is a conservation law and the corresponding conserved quantity will be called as Hamiltonian. The Hamiltonian which coincides with energy ($= KE + PE$) for a mechanical systems. For other system,qualifies to be identified with energy.

If Lagrangian does not contain t explicitly

$$\frac{\partial L}{\partial t} = 0 \tag{2}$$

$$\text{and} \qquad \frac{dL}{dt} = \sum_k \frac{\partial L}{\partial q_k}\dot{q}_k + \sum_k \frac{\partial L}{\partial \dot{q}_k}(\ddot{q}_k) \tag{3}$$

using Euler Lagrange EOM we get

$$\frac{dL}{dt} = \sum_k \Big[\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_k})\dot{q}_k + \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k\Big] \tag{4}$$

$$= \frac{d}{dt}\sum_{k=1} \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k \tag{5}$$

or

$$\frac{d}{dt}\Big(\sum_k \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L\Big) = 0 \tag{6}$$

Hence H defined by

$$H \stackrel{\text{def}}{\equiv} \sum_{k=1} \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L \tag{7}$$

is a constant of motion we can also write

$$H = \sum_{k=1}^{N} p_k\dot{q}_k - L \tag{8}$$

Where $p_k = \frac{\partial L}{\partial \dot{q}_k}$ is called the **canonical momentum conjugate** to the coordinate $\dot{q}_k$and H will be called **Hamiltonian** of the system

In an alternate of dynamics, the canonical momenta take over the role played by velocities and Hamiltonian becomes central quantity which governs the dynamics.The EOM can be written in an alternate form called the Hamiltonian EOM. In the Hamiltonian dynamics the velocities are eliminated in favour of canonical momenta.

An Example: Let us consider a single particle moving in force field described by potential energy $V(\vec{r})$.Then

$$L = \frac{1}{2}m\dot{\vec{r}}^2 - V(\vec{r}); \vec{r} = (x, y, z) \tag{9}$$

the canonical momenta are

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}; \qquad p_Y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}; \qquad p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \tag{10}$$

and the Hamiltonian is given by

$$\begin{aligned}
H &= \sum p_k \dot{q}_k - L & (11)\\
&= p_x m\dot{x} + p_y m\dot{y} + p_z m\dot{z} - L & (12)\\
&= m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - [\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(\vec{r})] & (13)\\
&= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V(\vec{r}) & (14)\\
&= \frac{1}{2}m\dot{\vec{r}}^2 + V(\vec{r}) & (15)
\end{aligned}$$

Thus the canonical momenta, in this example,coincide with components of momentum $m\dot{\vec{r}}$ and Hamiltonian is equal to the energy. However, it must be remarked that *the canonical momenta are not always equal to 'ordinary' momenta and Hamiltonian need not be a sum of $K.E + P.E$.*

This is the case when the system is described, for example, by a velocity dependent generalised potential. Motion of a charged particle in external magnetic field constitutes an example of this type where the canonical momentum is not equal to ordinary momentum.

§3 Points to remember

✍ The Hamiltonian is defined as

$$H = \sum_k p_k \dot{q}_k - L(q, \dot{(q)}, t) \qquad (16)$$

where p_k is *canonical momentum* conjugate to q_k. The canonical momentum is in general not the same as momentum of a particle.

✍ The Hamiltonian is just the total energy of a system. It is conserved When the Lagrangian does not have explicit time dependence.