Part-I From Newton to Euler and Lagrange

§ I-1 Euler Lagrange Equations of Motion

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§1 Prerequisites

1. We will start with Newton's laws of motion;
   In general for a physical system there may be *constraints* on the position vectors $\vec{r}_\alpha$ of the particles making up the system. Our discussion will be concerned

exclusively with holonomic constraints. These are given by equations of the form

$$\phi_k(\vec{r}_1, \ldots, \vec{r}_N, t) = 0, k = 1, \ldots$$

2. The forces of constraints have the property that the total work done by all forces of constraints on a system under a small virtual displacement is zero.

3. Principle of virtual work states that for a system in equilibrium work done by all forces under a *virtual displacement* is zero;

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§2 Constraints, Degrees of freedom, Generalized coordinates

The Lagrangian formulation of mechanics allows us to remove redundant coordinates and to deal with independent variables only. It also gives freedom to choose 'any' set of coordinates and use of Cartesian coordinates is not mandatory. Consider a system of N particles. Let $\vec{r}_\alpha, \alpha = 1 \ldots N$ denote the positions of the particles. We introduce generalised coordinates $q_k, k = 1, 2, \ldots$ by,

1. $q_1, q_2 \ldots q_n$ are functions of $\vec{r}_1, \vec{r}_2 \ldots$

2. All $q's$ are independent.

3. All $\vec{r}_\alpha$ are expressible in terms of $q_1, q_2, \ldots q_k \ldots$

The number of generalised coordinates q_k is equal to the number of **degrees of freedom**.

§3 From Newton's EOM to Euler Lagrange EOM

§3.1 Start with Newton's Second Law

Let $\vec{F}_\alpha$ be total force acting on α^{th} particle. The equations of motion are

$$\vec{F}_\alpha = \frac{d}{dt}\vec{p}_\alpha, \quad \alpha = 1, \ldots. \tag{1}$$

$\vec{p}_\alpha$ = momentum of α^{th} particle.

$$(or) \ \ (\vec{F}_\alpha - \frac{d}{dt}\vec{p}_\alpha) = 0 \tag{2}$$

For small displacements $\delta\vec{r}_\alpha$, we have

$$(\vec{F}_\alpha - \frac{d}{dt}\vec{p}_\alpha).\delta\vec{r}_\alpha = 0 \tag{3}$$

Summing over all particles

$$\sum_{\alpha=1}^{N}(\vec{F}_\alpha - \frac{d}{dt}\vec{p}_\alpha)\cdot\delta\vec{r}_\alpha = 0 \tag{4}$$

§3.2 Net force on a particle is a sum of external forces and forces due to constraint

We write total force $\vec{F}_\alpha$ as

$$\vec{F}_\alpha = \vec{F}_\alpha^{\ (e)} + \vec{f}_\alpha \tag{5}$$

Where $\vec{F}_\alpha^{(e)}$ is total external force on α^{th} particle and $\vec{f}_\alpha$ is the force due to constraint.

§3.3 Eliminating forces of constraints

Since the system is in equilibrium under forces of constraint, therefore virtual work principle gives

$$\sum \vec{f}_\alpha \cdot \delta\vec{r}_\alpha = 0. \tag{6}$$

Note that this important fact eliminates the forces of constraints.
Eqs.(4) - (6) give

$$\sum_{\alpha=1}^{N}(\vec{F}_\alpha^{(e)} - \frac{d}{dt}\vec{p}_\alpha)\cdot\delta\vec{r}_\alpha = 0 \tag{7}$$

where $\delta\vec{r}_\alpha =$ is virtual displacement of the system. This result (7) is known de Alembert's principle.

§3.4 Express 'total work done' in terms of generalised coordinates

The first term in Eq.(7), $\sum_{\alpha=1}^{N}(\vec{F}_\alpha^{(e)}\cdot\delta\vec{r}_\alpha)$ is the work done due to external forces. Next steps below are straightforward differential calculus manipulations. The strategy now is to introduce generalised coordinates $q_k, k = 1, 2, 3, \cdots$ and

- This result is known de Alembert's principle. to express $\delta\vec{r}_\alpha$ in terms of δq_k

$$\sum_\alpha ()\delta\vec{r}_\alpha \longrightarrow \sum_k ()\delta q_k$$

- to leave $\vec{v}_\alpha$, coming from $\vec{p}_\alpha$, alone. No attempt is made to change to $\dot{q}_k, q_k$ because we shall try to get final answer in terms of $T = \frac{1}{2}m_\alpha\vec{v}_\alpha^2$.

The Cartesian components of position vectors can be expressed as functions of $q_k, k = 1, 2, ...N$.

$$\vec{r}_\alpha = \vec{r}_\alpha(q_1, q_2 \ldots q_N, t) \tag{8}$$

The virtual displacements $\delta\vec{r}_\alpha$ in (7) are given by

$$\delta\vec{r}_\alpha = \sum_j \frac{\partial\vec{r}_\alpha}{\partial q_j}\delta q_j. \tag{9}$$

Remember that the coordinates q_k in $\vec{r}_\alpha$, see Eq.(8), change with time during the motion. Hence $\dfrac{\partial\vec{r}_\alpha}{\partial q_j}$ in (9), carry implicit time dependence. By adding and subtracting a suitable term, we rewrite the second term in expression (7) as

$$\Big\{\Big[\frac{d}{dt}(m_\alpha\vec{v}_\alpha)\Big]\Big[\frac{\partial\vec{r}_\alpha}{\partial q_j}\Big]\Big\}\delta q_j = \Big\{\frac{d}{dt}\Big[(m_\alpha\vec{v}_\alpha)\Big(\frac{\partial\vec{r}_\alpha}{\partial q_j}\Big)\Big] - m_\alpha v_\alpha\Big[\frac{d}{dt}\frac{\partial\vec{r}_\alpha}{\partial q_j}\Big]\Big\}\delta q_j \tag{10}$$

Differentiating (8) w.r.t. to time t we get

$$\frac{d\vec{r}_\alpha}{dt} = \sum_j \frac{\partial\vec{r}_\alpha}{\partial q_j}\dot{q}_j + \frac{\partial\vec{r}_\alpha}{\partial t}\Big|_{q_k}, \tag{11}$$

$$\text{or} \qquad \vec{v}_\alpha = \sum_j \frac{\partial\vec{r}_\alpha}{\partial q_j}\dot{q}_j + \frac{\partial\vec{r}_\alpha}{\partial t}. \tag{12}$$

Note that the velocities $\vec{v}_\alpha$ depend on $q, \dot{q}$ and t. Differentiating (12) w.r.t $\dot{q}_k$ we get

$$\frac{\partial\vec{v}_\alpha}{\partial\dot{q}_k} = \frac{\partial\vec{r}_\alpha}{\partial q_j}. \tag{13}$$

We shall use this relation to eliminate $\frac{\partial\vec{r}_\alpha}{\partial q_j}$ in the first term of (12).

Next note that

$$\frac{d}{dt}\Big(\frac{\partial\vec{r}_\alpha}{\partial q_j}\Big) = \frac{\partial}{\partial q_j}\Big(\frac{d\vec{r}_\alpha}{dt}\Big) = \frac{\partial\vec{v}_\alpha}{\partial q_j} \tag{14}$$

where differentiations w.r.t. t and $\vec{r}_\alpha$ have been exchanged in the last step. On using Eq.(12) and Eq.(14) in the right hand side, and summing over all particles Eq.(10), becomes

$$\sum_{j,\alpha}\frac{d}{dt}(m_\alpha v_\alpha)\Big(\frac{\partial\vec{r}_\alpha}{\partial q_j}\delta q_j\Big) \tag{15}$$

$$= \sum_{j,\alpha}\frac{d}{dt}(m_\alpha v_\alpha)\frac{\partial\vec{v}_\alpha}{\partial\dot{q}_j}\delta q_j \tag{16}$$

$$= \sum_{j,\alpha}\Big[\frac{d}{dt}\Big(m_\alpha\vec{v}_\alpha\frac{\partial\vec{v}_\alpha}{\partial\dot{q}_j}\Big) - m_\alpha\vec{v}_\alpha\frac{\partial\vec{v}_\alpha}{\partial q_j}\Big]\delta q_j \tag{17}$$

$$= \sum_{\alpha,j}\Big[\frac{1}{2}\frac{d}{dt}\frac{\partial}{\partial\dot{q}_k}(m_\alpha\vec{v}_\alpha^2) - \frac{1}{2}\frac{\partial}{\partial q_k}(m_\alpha\vec{v}_\alpha^2)\Big]\delta q_k \tag{18}$$

§3.5 Bring in kinetic energy

$$\sum_k \Big(\frac{d}{dt}\frac{\partial T}{\partial \dot{q_k}} - \frac{\partial T}{\partial q_k}\Big)\delta q_k \tag{19}$$

where T is the kinetic energy of the system

$$T = \sum_\alpha \frac{1}{2} m_\alpha \vec{v}_\alpha^2$$

Thus Eq(11), Eq(19) can be cast in the form

$$\sum \delta q_k \Big(\frac{d}{dt}\frac{\partial T}{\partial \dot{q_k}} - \frac{\partial T}{\partial q_k}\Big) = \sum_\alpha {F_\alpha}^{(e)} \cdot \frac{\partial \vec{r}_\alpha}{\partial q_k}\delta q_k \tag{20}$$

§3.6 Define Generalized force

The expression

$$\sum_\alpha {F_\alpha}^{(e)} \cdot \frac{\partial \vec{r}_\alpha}{\partial q_k} \equiv Q_k \tag{21}$$

will be called generalized force. In terms of generalized force, Eq.(20) becomes

$$\sum \delta q_k \Big(\frac{d}{dt}\frac{\partial T}{\partial \dot{q_k}} - \frac{\partial T}{\partial q_k}\Big) = Q_k \delta q_k \tag{22}$$

§3.7 The variations in generalized coordinates are independent

Calculus manipulations being over, it is time to use the fact that the generalised coordinates are independent
Since δq_k are independent and arbitrary the coefficient of each δq_k in the above equation can be set equal to zero.

$$\Big(\frac{d}{dt}\frac{\partial T}{\partial \dot{q_k}} - \frac{\partial T}{\partial q_k}\Big) = \sum_\alpha {F_\alpha}^{(e)} \frac{\partial \vec{r}_\alpha}{\partial q_k} \tag{23}$$

We will now consider two different special cases.

§4 Special Cases

§4.1 Case I: Conservative forces

Forces are conservative then there exists a function "V" called potential energy such that

$$\vec{F}_\alpha^{(e)} = -\vec{\nabla}_\alpha V \tag{24}$$

$$= -\Big(\frac{\partial V}{\partial x_\alpha}, \frac{\partial V}{\partial y_\alpha}, \frac{\partial V}{\partial z_\alpha}\Big) \tag{25}$$

Where $(x_\alpha, y_\alpha, z_\alpha) \equiv \vec{r}_\alpha$ are the components of position for the particle α. The r.h.s of (22) is given by

$$\sum_\alpha \vec{F}_\alpha \cdot \frac{\partial \vec{r}_\alpha}{\partial q_j} = -\sum_\alpha \Big(\frac{\partial v}{\partial x_\alpha}\frac{\partial x_\alpha}{\partial q_j} + \frac{\partial v}{\partial y_\alpha}\frac{\partial y_\alpha}{\partial q_j} + \frac{\partial v}{\partial z_\alpha}\frac{\partial z_\alpha}{\partial q_j} \Big) \tag{26}$$

$$= -\frac{\partial V}{\partial q_j} \tag{27}$$

Using (27) in (23) we get

$$\Big(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} \Big) = -\frac{\partial V}{\partial q_k} \tag{28}$$

Thus (28) can be rewritten as

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = \frac{d}{dt}\frac{\partial V}{\partial \dot{q}_k} - \frac{\partial V}{\partial q_k} \tag{29}$$

because V is a function of q's alone and $\frac{\partial V}{\partial \dot{q}_k} = 0$.

Introduce Lagrangian Therefore, rearranging the above equation we get

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0 \tag{30}$$

where $L = T - V$, is called the Lagrangian of the system. Eq.(30) are called Euler Lagrange equation of motion.

§4.2 Case II: Potential Dependent on Generalised Velocities

We assume that the forces depend on coordinates and velocities both and are such that they can be derived from a generalised potential U satisfying

$$\sum_\alpha \vec{F}^{(e)}_\alpha \frac{\partial \vec{r}_\alpha}{\partial q_j} = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_j} \tag{31}$$

where U is a function of $q, \dot{q}, t$

Then again (23) can be written in the Lagrangian form.

$$L = \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_j}\Big) - \frac{\partial L}{\partial q_j} = 0 \tag{32}$$

where we again have

$$L = T - U \tag{33}$$

We leave verification of (32) as a simple exercise for the reader. L is called Lagrangian and is a function of generalised coordinates q_j, generalised velocities $\dot{q}_j$ and t

$$L = L(q, \dot{q}, t) \tag{34}$$

We shall see that a description of motion of charges in electric and magnetic field requires use of a velocity dependent generalised potential recall that the magnetic forces are velocity dependent.

§5 Points to Remember

- ✍ For conservative systems, the Lagrangian formulation gets rid of constraints. There is no need of bringing in forces of constraints.

- ✍ For systems of many particles, the choice of generalised coordinates should be such that they are independent and the Cartesian coordinates $\vec{r}_\alpha$ for each particle can be expressed in terms of generalized coordinates.

- ✍ For conservative systems, the Lagrangian is given by $L = \text{K.E.} - \text{P.E.}$. The potential energy depends on q, $V = V(q)$.. The equations of motion now take the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0, \quad k = 1, \ldots N. \tag{35}$$

- ✍ For velocity dependent forces, we again have Euler Lagrange EOM the form (35) and

$$L = T - U$$

  where U is generalized potential depending on $q, \dot{q}$, $U = U(q, \dot{q})$.

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