# PY-402 Classical Mechanics Study Material
# Pages from Book by Calkin
# May 14 –July 6 (2018)

A.K. Kapoor

**These pages from Chapter II of the book by Calkin cover the prerequisites for PART-A**

# LAGRANGIAN
# AND
# HAMILTONIAN MECHANICS

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**LAGRANGIAN AND HAMILTONIAN MECHANICS**



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# PREFACE

This book grew out of notes for a one-semester course in classical mechanics which I have taught for many years to senior and, more recently, junior physics students at Dalhousie University. These students have normally taken one semester in mechanics at the introductory level and one semester at the second year level; my course is their final exposure to mechanics as undergraduates. The original aim of the course was to introduce the student, in the more familiar setting of classical mechanics, to those ideas and terms which he or she would later encounter in modified form in quantum mechanics: Lagrangian, action, Hamiltonian, Poisson brackets, canonical transformations, etc. In recent years, with the resurgence of interest in mechanics, especially non-linear dynamics, the emphasis of the course has shifted somewhat to include these contemporary developments as well.

The resulting book now contains more material than can be covered comfortably in a one-semester course. Experienced instructors can judge for themselves what material their students should omit on first reading; this depends on the level of the course and the emphasis the instructor wishes to place on the subject. While the book was written primarily as a fourth year text, the range of difficulty is quite broad. It can thus be used in third year, as I now do, simply by choosing the topics appropriately. Doing exercises forms an important part of any learning experience. The ones here have been chosen not only to exercise the students' minds, but also to provide additional examples and occasionally to introduce new topics.

Classical mechanics is a mature subject, and many others have written on it from their personal perspectives. I have indicated by footnotes throughout the book some of these other discussions which I have enjoyed studying and to which a student may refer to get a different point of view or a fuller treatment of some topic.

I wish to thank my students for encouraging me to organize my perspective of mechanics in this more formal way. My friend and colleague, David Kiang, has been most helpful. He read an early version of the manuscript and made many valuable suggestions for clarifying and improving the presentation and has been a continual source of advice on all aspects of the work. The support of my family is much appreciated. In particular, I thank my daughter Catherine for reading and editing my English; any smoothness in the writing is the result of her efforts; the roughness which remains, I take full responsibility for. My wife Patricia, good bow-paddler that she is, has helped me to avoid the rocks and shoals in this project, as in all aspects of our canoe trip together. I thank her from the bottom of my heart.

Melvin G. Calkin
Halifax, Nova Scotia
November, 1995

Pages From 1st Chapter are omitted

IMPORTANT POINTS, RESULTS etc NEEDED FOR OUR LECTURES ARE HIGHLIGHTED

# CHAPTER II

# THE PRINCIPLE OF VIRTUAL WORK AND D'ALEMBERT'S PRINCIPLE

In the previous chapter we saw how Newton's laws could be used directly to solve some simple mechanical problems involving point particles. We now turn to more general mechanical systems. We shall see that for most mechanical systems Newton's laws are *incomplete* and must be supplemented by additional conditions. These are contained in the principle of virtual work which is the subject of the present chapter.[1]

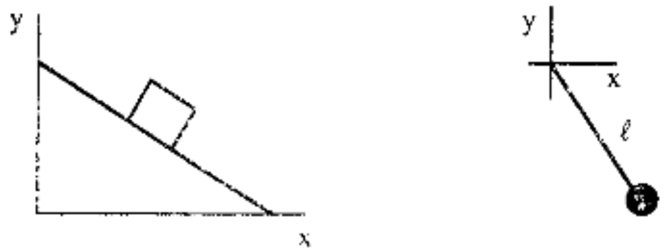
## Constraints

We begin by writing down Newton's second law as applied to a system of N particles,

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i \qquad i = 1, 2, \cdots, N.$$

At first it might appear that all we have to do is to integrate this coupled set of 3N equations to obtain the 3N coordinates $\mathbf{r}_i$ as functions of time. We soon discover, however, apart from the fact that the integration is unfeasible in most situations, that this set of equations is incomplete. There is more to mechanics than Newton's second law. In particular, the coordinates might be related or restricted by **constraints**. For example:

(a) The particles might be required to move on certain surfaces or curves, as for a block sliding on an inclined plane, or for a plane pendulum (Fig. 2.01).




The block moves on the surface $y = ax + b$

The bob moves on the curve $x^2 + y^2 = \ell^2$, $z = 0$

Fig. 2.01. Typical constraints

[1]For parallel reading see: Robert A. Becker, *Introduction to Theoretical Mechanics*, (McGraw-Hill Book Company, New York, NY, 1954), pp. 97-107; Cornelius Lanczos, *The Variational Principles of Mechanics*, (University of Toronto Press, Toronto, Ont., 1970; republished by Dover Publications, New York, NY, 1986), 4th ed., pp. 74-110; Arnold Sommerfeld, *Mechanics*, (Academic Press, New York, NY, 1952), trans. Martin O. Stern, pp. 48-66.

(b) The particles might be connected by "light rigid rods" so that the distances between them remain constant,

$$|\mathbf{r}_i - \mathbf{r}_j| = a_{ij},$$

as for the particles which make up a rigid body.

Constraints such as these which can be expressed as a set of equations of the form

$$G_l(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N; t) = 0 \qquad l = 1, 2, \cdots, 3N - f$$

are called **holonomic** constraints. The integer "f" is the number of degrees of freedom of the system. Other (non-holonomic) types of constraint, such as "particles confined to the interior of a box" or "wheel rolling over a surface," are difficult to handle in a general way and are not considered here.

Constraints have two effects:

1. The 3N coordinates $\mathbf{r}_i = (x_i, y_i, z_i)$ are not all independent. For a system with f degrees of freedom there are only f independent coordinates.

2. There are **forces of constraint** $\mathbf{F}_i^{(\text{constraint})}$ which the constraining surfaces, curves, rods, etc. exert on the particles so that they move in conformity with the constraints. These are initially *unknown* and must be found as part of the solution to the problem. If we call all the other forces **applied forces** and denote them by $\mathbf{F}_i^{(\text{applied})}$, the 3N equations arising from Newton's second law take the form

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i^{(\text{applied})} + \mathbf{F}_i^{(\text{constraint})} \qquad i = 1, 2, \cdots, N.$$

Together with the equations of constraint, these give a total of $6N - f$ equations for the 6N unknowns $\mathbf{r}_i$ and $\mathbf{F}_i^{(\text{constraint})}$. Thus at the moment we do not have sufficient information to solve the dynamical problem and must impose further conditions. In order to discover what these might be, let us look at some elementary examples where this situation occurs and see what we do in those cases.

## Principle of virtual work

First consider a block sliding on a frictionless incline near the surface of the earth (Fig. 2.02).

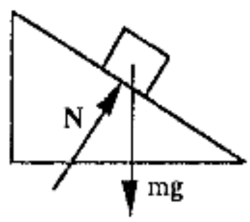



Fig. 2.02. Block on an incline

The block is subject to two forces: the force $mg$ gravity exerts on the block, an applied force, and the force $\mathbf{N}$ the incline exerts on the block, a force of constraint. If we think of this as a problem in two dimensions, there are four unknowns $x$, $y$, $N_x$, $N_y$. To find these, we have two equations from Newton's second law and one equation of constraint. The necessary fourth equation is the statement that $\mathbf{N}$ is perpendicular to the incline. We now wish to say this in a way which can be applied to a wide variety of situations. For this purpose we observe that another way to obtain the same result is to say that the force of constraint, being perpendicular to the displacement, does no work. We shall see that this idea, suitably extended, leads to the general additional condition we must impose on a mechanical system so as to make a well-set problem.

Next consider two particles connected by a light rigid rod and possibly subjected to external forces (Fig. 2.03).

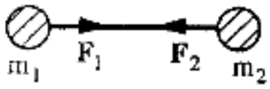



Fig. 2.03. Two interacting particles

We want to find the coordinates $\mathbf{r}_1$, $\mathbf{r}_2$ of the particles and the constraint forces $\mathbf{F}_1$, $\mathbf{F}_2$ the rod exerts on them, twelve unknowns in all. We have six equations from Newton's second law and one constraint equation. The remaining necessary equations are

$$\mathbf{F}_1 = -\mathbf{F}_2 \qquad \text{(3 equations)},$$

the force the rod exerts on particle 1 is equal and opposite the force it exerts on particle 2, and

"the forces are directed along a line joining the two particles" (2 equations).

How are we to summarize conveniently these requirements? We observe that for any displacement of the system, while the forces of constraint $\mathbf{F}_1$ and $\mathbf{F}_2$ may do work individually, the *net* work

$$\delta W = \mathbf{F}_1 \cdot \delta\mathbf{r}_1 + \mathbf{F}_2 \cdot \delta\mathbf{r}_2$$

done by all the forces of constraint is again zero. To see this, note that the displacements are of two types: *translations*, for which $\delta\mathbf{r}_1 = \delta\mathbf{r}_2$, and $\delta W = 0$ because $\mathbf{F}_1 = -\mathbf{F}_2$ and the work done by $\mathbf{F}_1$ is equal and opposite the work done by $\mathbf{F}_2$; and *rotations*, for which the displacements are perpendicular to the line joining the two particles, and the work done by $\mathbf{F}_1$ and $\mathbf{F}_2$ are each zero because the forces lie along the line joining the two particles. Further, by reversing the argument we can obtain the preceding five conditions on the forces of constraint from the statement about work; they are equivalent.

As we continue to examine a wide variety of situations, we may be tempted to summarize our observations by saying "the net work done by forces of constraint is zero," but this is not quite true. Forces of constraint *can* do work if the constraint is time-dependent, if the incline in the first example is moving or the length of the rod in the second example is changing. Consider Fig. 2.04,

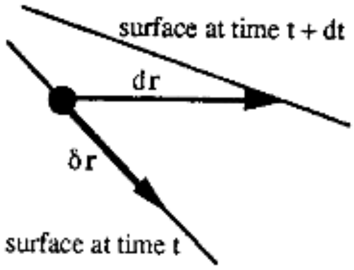



Fig. 2.04. Real and virtual displacements for a time-varying constraint

which shows a particle constrained to a surface. If the surface moves in the time interval $t$ to $t + dt$, the real displacement $d\mathbf{r}$ of the particle has a component normal to the surface, in the direction of the force of constraint, so the force of constraint in this situation can do work. Thus in order to apply our work idea we must modify our prescription as follows: *freeze* the system at some instant of time $t$; then *imagine* the particles displaced amounts $\delta\mathbf{r}_i$ consistent with the conditions of constraint. This is called a **virtual displacement**. We use $\delta\mathbf{r}_i$ rather than $d\mathbf{r}_i$ to distinguish virtual displacements from real displacements. We then apply our work idea not to real displacements but to virtual displacements, stating the result as follows:

**The principle of virtual work**

The total work done by the forces of constraint in a virtual displacement is zero,

$$\delta W^{(\text{constraint})} = \sum_{i=1}^{N} \mathbf{F}_i^{(\text{constraint})} \cdot \delta\mathbf{r}_i = 0.$$

(虚功即可代表
约束的条件)

We have not given a "proof" of the principle of virtual work, but rather an indication of some types of situation in which the principle holds. Readers will have to judge from physical considerations whether and in what sense the principle holds for the particular physical system they wish to consider. The principle is in a sense a statement about what forces we can consider "forces of constraint," and it summarizes their properties. Forces which do not satisfy the principle must be considered "applied forces."

As we now show, the principle of virtual work provides the additional $f$ equations needed, besides the $3N$ from Newton's second law and the $3N - f$ equations of constraint, to complete the specification of the dynamical problem. First suppose that there are no constraints. Then all the $\delta\mathbf{r}_i$'s are independent, and the only way $\delta W^{(\text{constraint})}$ can be zero for all $\delta\mathbf{r}_i$ is if $\mathbf{F}_i^{(\text{constraint})} = 0$; these $3N$ equations say, correctly, that in this case there are no forces of constraint. Now suppose that there is one constraint. The coordinates are then connected by one equation of the form

$$G(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N; t) = 0,$$

and the number of degrees of freedom is $3N-1$. That is, $3N-1$ of the $\mathbf{r}_i = (x_i, y_i, z_i)$ are independent, and 1 is dependent. If we express this one dependent variable in terms of the independent variables $x_j^{(ind)}$, the principle of virtual work gives

$$\sum_{j=1}^{3N-1}\left(\sum_{i=1}^{N}\mathbf{F}_i^{(constraint)}\cdot\frac{\partial\mathbf{r}_i}{\partial x_j^{(ind)}}\right)\delta x_j^{(ind)} = 0 .$$

The coefficient of each of the $\delta x_j^{(ind)}$ must vanish, giving $3N-1$ restrictions on the $\mathbf{F}_i^{(constraint)}$'s, as required. It is clear that this argument can be easily generalized to the case where there are $3N-f$ equations of constraint and f independent variables. Each time we add a constraint equation we reduce the number of degrees of freedom, the number of independent variables, by one and hence reduce the number of conditions on the $\mathbf{F}_i^{(constraint)}$ by one; the number of constraint equations plus conditions on the $\mathbf{F}_i^{(constraint)}$ remains fixed at 3N. To summarize, the principle of virtual work provides the additional equations needed to make a well-set mechanical problem.

## D'Alembert's principle and generalized coordinates

Quite often we are not interested in the forces of constraint themselves. We can then use Newton's second law to eliminate them from the remaining equations, setting

$$\mathbf{F}_i^{(constraint)} = m_i\ddot{\mathbf{r}}_i - \mathbf{F}_i^{(applied)}$$

in the principle of virtual work. We are left with

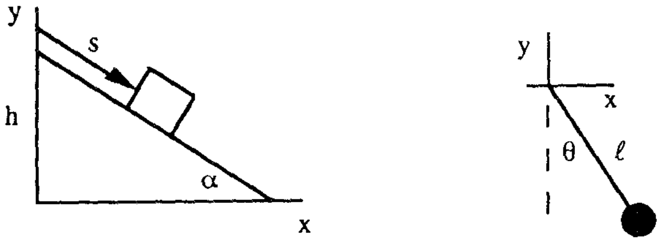
$$\sum_{i=1}^{N}\left(\mathbf{F}_i^{(applied)} - m_i\ddot{\mathbf{r}}_i\right)\cdot\delta\mathbf{r}_i = 0 .$$

This is **d'Alembert's principle**. It says that the work done by the applied forces, plus the work done by the so-called **inertial forces** $-m_i\ddot{\mathbf{r}}_i$, in a virtual displacement is zero. In spite of its simple appearance, d'Alembert's principle is extremely important. Together with the equations of constraint, it *contains* Newton's second law as well as the necessary conditions on the forces of constraint. It forms the basis for all our further developments in mechanics.

Rather than using a set of 3N *non-independent* variables $\mathbf{r}_i$ which are connected by the $3N-f$ equations of constraint, it is more convenient to use a set of $f$ $(\le 3N)$ *independent* variables $q_a$ $(a = 1,2,\cdots,f)$, the **generalized coordinates**, to describe the configuration of the system. We have great freedom in the choice of these coordinates. Essentially any set of f independent variables from which the $\mathbf{r}_i$ can be obtained by equations of the form

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \cdots, q_f; t) \qquad i = 1, 2, \cdots, N$$

will serve. See, for example, Fig. 2.05.




(a) For the block on an inclined plane the horizontal coordinate x *or* the vertical coordinate y *or* the distance s down the plane would all serve as the generalized coordinate. In terms of the latter variable the cartesian coordinates are

$x = s\cos\alpha \quad y = h - s\sin\alpha$

(b) For the plane pendulum the horizontal coordinate x *or* the vertical coordinate y *or* the angle θ would all serve as the generalized coordinate. In terms of the latter variable the cartesian coordinates are

$x = \ell\sin\theta \quad y = -\ell\cos\theta$

Fig. 2.05. Typical generalized coordinates

Once we have introduced generalized coordinates for a system, the dynamics is completely contained in d'Alembert's principle. Let us see how to use it for some simple problems in mechanics.

## Lever

A (horizontal) lever is in static equilibrium under the application of (vertical) forces $F_1$ a distance $\ell_1$ from the fulcrum, and $F_2$ a distance $\ell_2$ from the fulcrum, as shown in Fig. 2.06. 支点

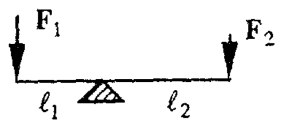



Fig. 2.06. Lever

What is the condition on these quantities for equilibrium to obtain? Although the answer is well-known to any child who has experimented with a teeter-totter, obtaining it via d'Alembert's principle is instructive. To do this, imagine the lever to undergo a virtual displacement, a (say) clockwise rotation about its fulcrum through an infinitesimal angle

$\delta\theta$. End 1 moves up a distance $\ell_1\delta\theta$ and the applied force $F_1$ does work $-F_1\ell_1\delta\theta$; end 2 moves down a distance $\ell_2\delta\theta$ and the applied force $F_2$ does work $+F_2\ell_2\delta\theta$. In this problem of static equilibrium there are no inertial forces, so d'Alembert's principle yields

$$-F_1\ell_1\delta\theta + F_2\ell_2\delta\theta = 0,$$

which gives the well-known condition

$$F_1\ell_1 = F_2\ell_2.$$

## Inclined plane

A block of mass m slides on an inclined plane under the influence of gravity. We take as generalized coordinate the displacement s down the plane (Fig. 2.07).

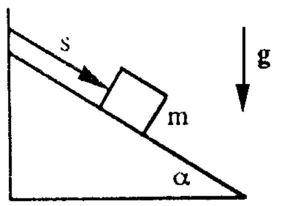



Fig. 2.07. Inclined plane

What is the equation of motion? To apply d'Alembert's principle, imagine that the block undergoes a virtual displacement $\delta s$ down the plane. The applied force, gravity, does work $mg\sin\alpha\,\delta s$. The acceleration of the block down the plane is $\ddot{s}$, so the inertial force on it is $m\ddot{s}$ up the plane, and the work done by the inertial force is $-m\ddot{s}\delta s$. D'Alembert's principle then says

$$mg\sin\alpha\,\delta s - m\ddot{s}\delta s = 0,$$

which yields the well-known result

$$\ddot{s} = g\sin\alpha.$$

## Plane pendulum

A bob of mass m is suspended from the ceiling by a string[2] of length $\ell$ and can swing back and forth in a vertical plane under the influence of gravity g. The system has

[2]Although the word "string" is used here and in other similar situations throughout the text, the phrase "light rigid rod" is sometimes meant.

one degree of freedom, and we can take as generalized coordinate the angular displacement $\theta$ from vertical (Fig. 2.08).

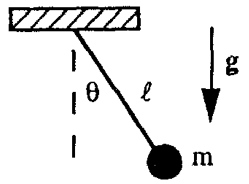



Fig. 2.08. Plane pendulum

What is the equation of motion? To apply d'Alembert's principle, imagine a virtual displacement in which $\theta$ increases by a small amount $\delta\theta$. The bob rises a distance $\ell\delta\theta\sin\theta$, and the only applied force, gravity, does work $-(mg)(\ell\delta\theta\sin\theta)$. The acceleration of the bob in the direction of the virtual displacement is $\ell\ddot{\theta}$, so the work done by the inertial force is $(-m\ell\ddot{\theta})(\ell\delta\theta)$. D'Alembert's principle then gives

$$(-mg)(\ell\delta\theta\sin\theta) - (m\ell\ddot{\theta})(\ell\delta\theta) = 0,$$

which simplifies to

$$\ddot{\theta} = -(g/\ell)\sin\theta.$$

This is the required equation of motion.

Now suppose that the length of the supporting string is time-dependent; perhaps it is expanding and contracting with changes in temperature. A virtual displacement at time t is the same as before, a distance $\ell(t)\delta\theta$ in the $\theta$-direction, so the work done by gravity is the same. But now the acceleration of the bob has a component $\ell\ddot{\theta} + 2\dot{\ell}\dot{\theta}$ in the $\theta$-direction, so the work done by the inertial force is $-m(\ell\ddot{\theta} + 2\dot{\ell}\dot{\theta})\ell\delta\theta$. D'Alembert's principle gives

$$-(mg)(\ell\delta\theta\sin\theta) - m(\ell\ddot{\theta} + 2\dot{\ell}\dot{\theta})\ell\delta\theta = 0,$$

which yields

$$\frac{d}{dt}(m\ell^2\dot{\theta}) = -mg\ell\sin\theta.$$

The quantity $m\ell^2\dot{\theta}$ is the angular momentum of the bob about the point of support. If $g = 0$, in which case the plane pendulum becomes a plane rotator, it remains constant even if $\ell$ changes with time (in contrast to, say, the kinetic energy).

Another way in which the length of the pendulum could change with time would be for the string to pass through a small hole in the ceiling and be acted on by a force F (Fig. 2.09).

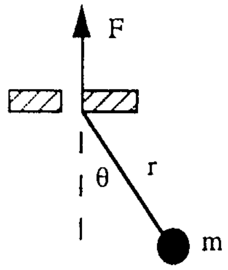



Fig. 2.09. Plane pendulum with time-varying length

The system now has *two* degrees of freedom, and we can take as generalized coordinates the angle $\theta$ and the length r of the pendulum (replacing $\ell$). There are two independent virtual displacements:

(a) Vary $\theta$, keeping r fixed. This is the same as we had in the previous paragraph, and d'Alembert's principle yields in the new notation

$$\frac{d}{dt}(mr^2\dot{\theta}) = -mgr\sin\theta .$$

(b) Vary r, keeping $\theta$ fixed. Imagine increasing r an amount $\delta r$. The applied force gravity does work $mg\delta r\cos\theta$. The applied force F does work $-F\delta r$. The acceleration of the bob has a component $\ddot{r} - r\dot{\theta}^2$ in the r-direction, so the work done by the inertial force is $-m(\ddot{r} - r\dot{\theta}^2)\delta r$. D'Alembert's principle gives

$$mg\delta r\cos\theta - F\delta r - m(\ddot{r} - r\dot{\theta}^2)\delta r = 0,$$

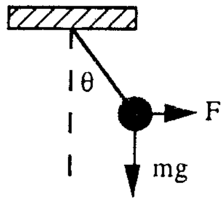
which yields

$$m(\ddot{r} - r\dot{\theta}^2) = -F + mg\cos\theta .$$

For $g = 0$ these are simply the general equations for motion under a central force F.

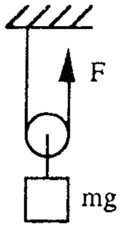
## Exercises

1.



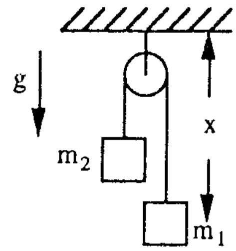

Use d'Alembert's principle to find the condition of static equilibrium.

2.



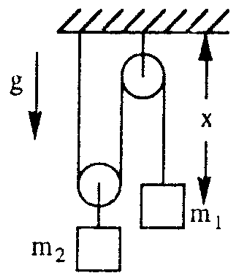

Use d'Alembert's principle to find the condition of static equilibrium.

3.



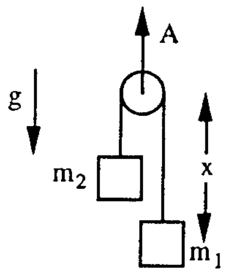

Use d'Alembert's principle to find the acceleration of $m_1$.

4.



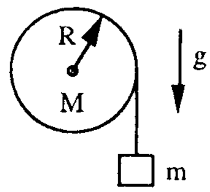

Use d'Alembert's principle to find the acceleration of $m_1$.

5.



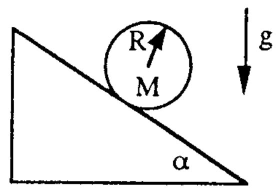

Use d'Alembert's principle to find the acceleration of $m_1$. Note that in this case the pulley has an upward acceleration A. "Acceleration" means "acceleration relative to the earth."

6.



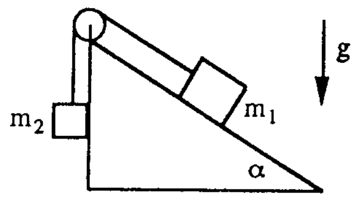

A mass m is attached to a light cord which wraps around a frictionless pulley of mass M, radius R, and moment of inertia $I = \int r^2 dM$. Gravity g acts vertically downwards. Use d'Alembert's principle to find the acceleration of m.

7.



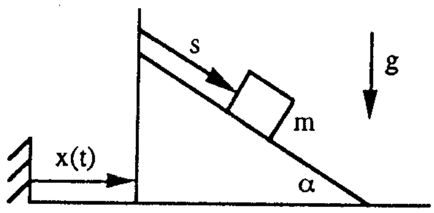

A cylinder of mass M, radius R, and moment of inertia $I = \int r^2 dM$ rolls without slipping down an inclined plane. Use d'Alembert's principle to find the acceleration of the cylinder.

8.



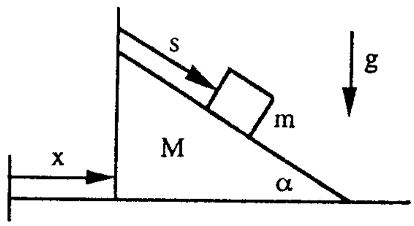

Use d'Alembert's principle to find the acceleration of $m_1$ down the (stationary) plane.

9.




A block of mass m slides on a frictionless inclined plane, which is driven so that it moves horizontally, the displacement of the plane at time t being some known function x(t). Use d'Alembert's principle to find the equation of motion of the block, taking as generalized coordinate the displacement s of the block down the plane. Note that the acceleration of the block is *not* "down the plane."

10.




A block of mass m slides on a frictionless inclined plane of mass M which in turn is free to slide on a frictionless horizontal surface. Use d'Alembert's principle to find the equations of motion of the block and the plane, taking as generalized coordinates the displacement s of the block down the plane and the horizontal displacement x of the plane.