Phy 523 Problem sheet IX (SOLUTIONS)

41. Consider the S- matrix element of Compton scattering $\gamma(k_1)+e(p_1) \rightarrow \gamma(k_2)+e(p_2)$ in the form

$$N\epsilon^{\mu}(k_1)\epsilon^{*\nu}(k_2)M_{\mu\nu}(2\pi)^4\delta^4(k_1+p_1-k_2-p_2)$$

. where N is the normalisation constant.

Show that $k_1^{\mu}\epsilon^{*\nu}M_{\mu\nu} = k_2^{\nu}\epsilon^{\mu}M_{\mu\nu} = 0$. (This proves the amplitude is gauge invariant: In momentum space, $\epsilon^{\mu}(k)$ changes to $\epsilon^{\mu}+\alpha k^{\mu}$ under a gauge transformation. Here α is the gauge parameter.)

Solution:

We have to show that $N\epsilon^{\mu}(k_1)\epsilon^{*\nu}(k_2)M_{\mu\nu}(2\pi)^4\delta^4(k_1+p_1-k_2-p_2).= N(\epsilon^{\mu}(k_1)+\alpha k_1^{\mu})(\epsilon^{*\nu}(k_2)+\alpha^{*}k_2^{\mu})M_{\mu\nu}(2\pi)^4\delta^4(k_1+p_1-k_2-p_2)$ We consider the term

$$k_1^{\mu}\epsilon^{*\nu}M_{\mu\nu} = k_1^{\mu}e^2[\bar{u}(p_2)\not{\epsilon}^{*}(k_2)\frac{i}{\not{p}_1+\not{k}_1-m}\gamma_{\mu}u(p_1)$$

$$+\bar{u}(p_2)\gamma_{\mu}\frac{i}{\not{p}_2-\not{k}_1-m}\not{\epsilon}^{*}(k_2)u(p_1)]$$

$$= e^2[\bar{u}(p_2)\not{\epsilon}^{*}(k_2)\frac{i}{\not{p}_1+\not{k}_1-m}\not{k}_1u(p_1)+\bar{u}(p_2)\not{k}_1\frac{1}{\not{p}_2-\not{k}_1-m}\not{\epsilon}^{*}(k_2)u(p_1)]$$

We write $\not{k}_1$ occuring in the first term as $(\not{p}_1+\not{k}_1-m)$ as $(\not{p}_1-m)u(p_1)=0$. Similarly we write k_1 as $-\not{p}_2+\not{k}_1+m$ for the second term in place of just $\not{k}_1$ (now the identity used is $\bar{u}(p_2)(-\not{p}_2+m)=0$.) This gives

$$= e^2[\bar{u}(p_2)\not{\epsilon}^{*}(k_2)\frac{i}{\not{p}_1+\not{k}_1-m}(\not{k}_1+\not{p}_1-m)u(p_1)$$

$$+\bar{u}(p_2)(m-\not{p}_2+\not{k}_1)\frac{1}{\not{p}_2-\not{k}_1-m}\not{\epsilon}^{*}(k_2)u(p_1)]$$

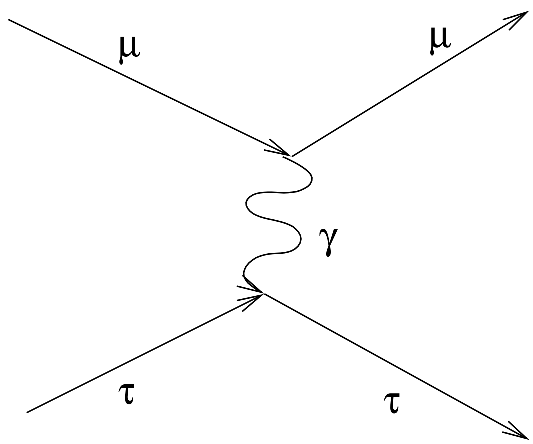
The factors $\not{p}_1+\not{k}_1-m$ in the numerator and the denominator cancel in the first term. Similarly the factor $m+\not{k}_1-\not{p}_2$ cancels with the factor $\not{p}_2-\not{k}_1-m$ in the denominator giving a factor -1. Thus

$$= e^2[\bar{u}(p_2)\not{\epsilon}^{*}(k_2)(i)u(p_1)-\bar{u}(p_2)\not{\epsilon}^{*}(k_2)(i)u(p_1)] = 0$$

Similarly one can show that $k_2^\nu M_{\mu\nu} = 0$. This completes the proof that the amplitude is gauge invariant.

42. Write the matrix element for $\tau^- + \mu^+ \to \tau^- + \mu^+$ assuming only electromagnetic interaction for τ-lepton and muon.

Solution:



Feynman diagram for $\mu(p_1) + \tau(q_1) \to \mu(p_2) + \tau(q_2)$ The matrix element is

$$\frac{-e^2}{(16p_1^0 p_2^0 q_1^0 q_2^0)^{1/2}} \bar{u}_\mu(p_2)\gamma^\alpha u_\mu(p_1) \frac{-i\eta_{\alpha\beta}}{(p_1 - p_2)^2} \bar{u}_\tau(q_2)\gamma^\beta u_\tau(q_1)$$
$$(2\pi)^4\delta^4(p_1 + q_1 - p_2 - q_2)$$

43. τ-lepton has a mass of about 1.8 GeV. τ^- can decay to $\pi + \nu_\tau$. Using the expression of $\pi^- \to \mu^- + \bar{\nu}_\mu$ decay write the matrix element for $\tau^- \to \pi + \nu_\tau$decay.

Solution:

We write for $\pi^-(P)$-decay to $\mu^-(q_1) + \bar{\nu}_\mu(q_2)$ the matrix element, ignoring the overall phase factor,

$$M = \frac{1}{(4q_1^0 q_2^0)^{1/2}}(2\pi)^4\delta^4(P - q_1 - q_2)\bar{u}_\mu(q_1)\gamma^\beta(1 - \gamma_5)v_{\nu_\mu}(q_2) < 0|J_\beta|\pi >$$

where J_β is the current and

$$< 0|J_\beta|\pi >= \frac{F_\pi}{(2P^0)^{1/2}} P_\beta$$

. For the $\tau(\bar{q}_1)$-decay to $\pi(\bar{P}) + \nu_\tau(\bar{q}_2)$ we have

$$M = \frac{1}{(4\bar{q}_1^0\bar{q}_2^0)^{1/2}}(2\pi)^4\delta(-\bar{P} + \bar{q}_1 - \bar{q}_2)\bar{u}_{\nu_\tau}(\bar{q}_2)\gamma^\beta(1-\gamma_5)u_\tau(\bar{q}_1) < \pi(\bar{P})|J_\beta^\dagger|0 >$$

here

$$< \pi(\bar{P})|J_\beta^\dagger|0 >=< 0|J_\beta|\pi(\bar{P}) >^*= \frac{F_\pi^*}{(2\bar{P}^0)^{1/2}}\bar{P}_\beta$$

44. Calculate the decay rate for the τ-decay in the problem 43.

Solution:
The matrix element becomes

$$M(\tau \to \pi + \nu_\tau) = \frac{F_\pi^*}{(8\bar{P}^0\bar{q}_1^0\bar{q}_2^0)^{1/2}}\bar{u}_{\nu_\tau}(\bar{q}_2) \not{\bar{P}} u_\tau(\bar{q}_1)(2\pi)^4\delta^4(\bar{q}_1 - \bar{P} - \bar{q}_2)$$

Using the relation $\bar{P} = \bar{q}_1 - \bar{q}_2$ and $\bar{u}_{\nu_\tau}(q_2) \not{\bar{q}}_2 = 0$, $\not{\bar{q}}_1(1-\gamma_5)u_\tau(\bar{q}_1) = M_\tau(1+\gamma_5)u_\tau(\bar{q}_1)$ we have

$$M = M_\tau \frac{F_\pi^*}{(8\bar{P}^0\bar{q}_1^0\bar{q}_2^0)^{1/2}}\bar{u}_{\nu_\tau}(\bar{q}_2)(1+\gamma_5)u_\tau(\bar{q}_1)(2\pi)^4\delta^4(\bar{q}_1 - \bar{P} - \bar{q}_2)$$

Squaring the amplitude

$$\frac{|M|^2}{VT} = M_\tau^2|F_\pi|^2\frac{(2\pi)^4\delta(\bar{q}_1 - \bar{P} - \bar{q}_2)}{8\bar{P}^0\bar{q}_1{}^0\bar{q}_2{}^0}|\bar{u}_{\nu_\tau}(\bar{q}_2)(1+\gamma_5)u_\tau(\bar{q}_1)|^2$$

We sum over the final spins and average over the initial spins to get

$$\frac{\sum_{spins}|\bar{M}|^2}{VT} = M_\tau^2|F_\pi|^2\frac{(2\pi)^4\delta(\bar{q}_1 - \bar{P} - \bar{q}_2)}{8\bar{P}^0\bar{q}_1{}^0\bar{q}_2{}^0}\frac{1}{2}Tr(\bar{q}_2(1+\gamma_5)(\bar{q}_1 + M_\tau)(1-\gamma_5))$$

$$= M_\tau^2|F_\pi|^2\frac{(2\pi)^4\delta(\bar{q}_1 - \bar{P} - \bar{q}_2)}{8\bar{P}^0\bar{q}_1{}^0\bar{q}_2{}^0}\frac{1}{2}8\bar{q}_1.\bar{q}_2$$

Using $\bar{q}_1.\bar{q}_2 = M_\tau\bar{q}_2{}^0$ in the rest frame of τ we get for the decay rate

$$\Gamma(\tau \to \pi + \nu_\tau) = \int \frac{d^3\bar{P}d^3\bar{q}_2}{(2\pi)^6}M_\tau^2|F_\pi|^2\frac{(2\pi)^4\delta(\bar{q}_1 - \bar{P} - \bar{q}_2)}{8\bar{P}^0\bar{q}_1{}^0\bar{q}_2{}^0}\frac{1}{2}8M_\tau\bar{q}_2{}^0$$

$$= \int M_\tau^2 |F_\pi|^2 \frac{d^3\bar{P}}{(2\pi)^2} \frac{\delta(\bar{q_1}^0 - \bar{P}^0 - \bar{q_2}^0)}{8\bar{P}^0 \bar{q_1}^0 \bar{q_2}^0} 4M_\tau \bar{q_2}^0$$

we can perform the angular integral of $d^3\bar{P} = d\omega_{\bar{P}} \bar{P}^2 d\bar{P}$ as there is no angular dependence on this variable and write

$$= \int 4\pi M_\tau^2 |F_\pi|^2 \frac{\bar{P}^2 d\bar{P}}{(2\pi)^2} \frac{\delta(\bar{q_1}^0 - \bar{P}^0 - \bar{q_2}^0)}{8\bar{P}^0 \bar{q_1}^0 \bar{q_2}^0} 4M_\tau \bar{q_2}^0$$

The integral over $\bar{P}$can be done using the δ-function. Remembering $\bar{q}_2^0 = \bar{P}, \bar{P}^0 = (m_\pi^2 + \bar{P}^2)^{1/2}$, we get a factor $1 + \bar{P}/\bar{P}^0 = M_\tau/\bar{P}^0$ in the denominator after the integration, leading to

$$= 4\pi M_\tau^2 |F_\pi|^2 \frac{\bar{P}^2}{(2\pi)^2} \frac{1}{8\bar{P}^0 \bar{q_1}^0 \bar{q_2}^0} \frac{1}{\frac{M_\tau}{\bar{P}^0}} 4M_\tau \bar{q_2}^0$$

This leads to

$$\Gamma = \frac{|F_\pi|^2 M_\tau}{2\pi} |\bar{P}|^2.$$

45. Write possible three body leptonic decays of τ-lepton and the corresponding matrix element assuming the interaction is given by the standard model.

Solution:

The possible three body leptonic decays are

$$\tau^-(P) \to \mu^-(Q) + \nu_\tau(q_1) + \bar{\nu}_\mu(q_2)$$

and

$$\tau^-(P) \to e^-(Q) + \nu_\tau(q_1) + \bar{\nu}_e(q_2)$$

The matrix elements are

$$M_1 = N\bar{u}_{\nu_\tau}(q_1)\gamma^\alpha(1-\gamma_5)u_\tau(P)(-i)\left(\frac{\eta_{\alpha\beta} - \left(\frac{r_\alpha r_\beta}{m_W^2}\right)}{r^2 - m_W^2}\right)\bar{u}_\mu(Q)\gamma^\beta(1-\gamma_5)v_{\nu_\mu}(q_2)$$

and

$$M_2 = N\bar{u}_{\nu_\tau}(q_1)\gamma^\alpha(1-\gamma_5)u_\tau(P)(-i)\left(\frac{\eta_{\alpha\beta} - \left(\frac{r_\alpha r_\beta}{m_W^2}\right)}{r^2 - m_W^2}\right)\bar{u}_e(Q)\gamma^\beta(1-\gamma_5)v_{\mu_e}(q_2)$$

Here

$$N = \frac{(2\pi)^4 \delta^4(P - Q - q_1 - q_2)}{16 P^0 Q^0 q_1^0 q_2^0} \frac{g^2}{8}$$

and $r = P - q_1$