Phy 523
PARTICLE PHYSICS
SOLUTIONS: PROBLEM SHEET VII

31. Consider the spin zero particle under the action of a scalar current J obeying the equation

$$\partial^\mu \partial_\mu \Phi(x) + m^2 \Phi(x) = -J(x)$$

Writing the Greens function as

$$(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} + m^2)\Delta_F(x,y) = -\delta^4(x-y)$$

show that the general solution of the spin one equation in the presence of J is

$$\Phi_i(x) = \phi_i^0 + \int d^4z \Delta_F(x-z)J(z)$$

where ϕ_i^0 is the solution of the free Klein-Gordon equation.

Solution:
We have

$$(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} + m^2)\Phi(x) = (\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} + m^2)\phi_i^0 + \int d^4z (\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} + m^2)\Delta(x-z)J(z)$$

$$= -\int d^4z \delta^4(x-z)J(z) = -J(x)$$

as the first termis zero because ϕ_i^0 obeys the free particle Klein-Gordon equation

32. Evaluate the propator $\Delta_F(x-y)$ using the Feynman boundary condition and show it can be written as

$$\Delta_F(x-y) = -i\theta(x_0 - y_0)\int \frac{d^3p}{(2\pi)^3} f_p^+(x) f_p^{(+)*}(y)$$

$$-i\theta(y_0 - x_0)\int \frac{d^3p}{(2\pi)^3} f_p^-(x) f_p^{(-)*}(z)$$

. where

$$f_p^+(x) = \frac{1}{\sqrt{2p^0}}e^{-ip.x}; \quad f_p^-(x) = \frac{1}{\sqrt{2p^0}}e^{ip.x}$$

Soultion:

We start by going to the momentum space

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\Delta_F(p)e^{-i(p.(x-y)}$$

Using the definition of $\Delta_F(x-y)$ from Prob.31 we have using the fourier transform of δ-function

$$(-p^2+m^2)\Delta_F(p) = -1$$

or

$$\Delta(p) = \frac{1}{p^2-m^2}$$

this gives

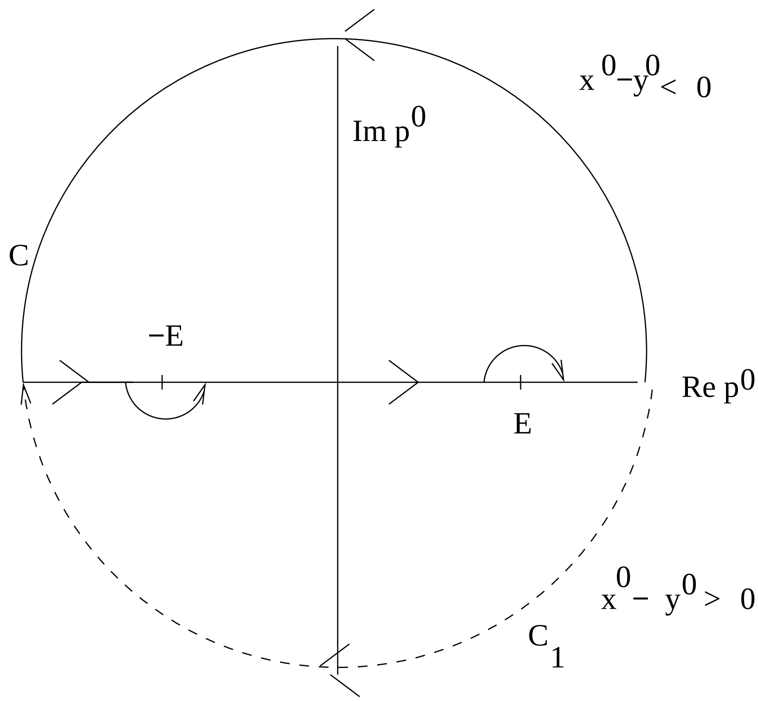
$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{-i(p.(x-y)}}{p^2-m^2}$$

The integral over p^0 has a pole at $\pm(|\vec{p}|^2+m^2)^{1/2}$ and the boundary conditions decide on the contour to be chosen. We want positive energy solutions to propagate forwards in time and negative energy solutions to propagare backwards in time. This can be done by using an approriate contour C in the complex p^0 plane as shown in the figure. If $(x^0-y^0)>0$ then we use the contour C_1i(dashed), closing it from the lower half plane. The contribution of the exponential term ensures that the semi circular part's contribution is zero for $p^0(x^0-y^0)$ has a negative imaginary part $(Imp^0, 0)$ and the $e^{-ip^0(x^0-y^0)} \to 0$ as the radius of the semi circle $\to \infty$. On doing the p^0 integration only the pole $p^0 = E$ contributes and we get

$$\theta(x^0-y^0)\Delta_F(x-y) = \int \frac{d^3p}{(2\pi)^4}e^{-iE(x^0-y^0)+i\vec{p}.(\vec{x}-\vec{y})}\frac{(-2\pi i)}{2E}$$

$$= -i\int \frac{d^3p}{(2\pi)^3}\frac{e^{-iE(x^0-y^0)+i\vec{p}.(\vec{x}-\vec{y})}}{2E}$$

as the contour C_1 is clockwise and the residue at the pole $p^0 = E$ is $2E$



For $x^0 - y^0 < 0$ we get (pole at $-E$, residue is $-2E$ and the contour C is counter clockwise)

$$\theta(-x^0 + y^0)\Delta_F(x-y) = -i\int \frac{d^3p}{(2\pi)^3}\frac{e^{+iE(x^0-y^0)+i\vec{p}.(\vec{x}-\vec{y})}}{2E}$$

$$= -i\int \frac{d^3p}{(2\pi)^3}\frac{e^{+iE(x^0-y^0)-i\vec{p}.(\vec{x}-\vec{y})}}{2E}$$

where we have changed the variables of integration over the three momentum from $\vec{p}$ to $-\vec{p}$. We can now write the expression for the general case using

$$f_p^+(x) = \frac{1}{\sqrt{2p^0}} e^{-ip.x}; \quad f_p^-(x) = \frac{1}{\sqrt{2p^0}} e^{ip.x}$$

$$\Delta_F(x-y) = -i\theta(x_0 - y_0) \int \frac{d^3p}{(2\pi)^3} f_p^+(x) f_p^{(+)*}(y)$$
$$-i\theta(y_0 - x_0) \int \frac{d^3p}{(2\pi)^3} f_p^-(x) f_p^{(-)*}(y)$$

33. Let $J(x) = g\bar{\psi}(x)\psi(x)$ where is a spin half field and g is the coupling constant. Ψ obeys the equation

$$(i \not{\partial} - m)\Psi(x) = -g\phi(x)\Psi(x)... \quad Eq.(1)$$

. Obtain the expression for the S-matrix element

$$S_{fi} = \delta_{fi} + ig \int d^4y \bar{\psi}(y)\phi(y)\Psi(y)$$

where $\Psi(x)$ is the solution of Eq.(1) and can be written as

$$\Psi(x) = \psi(x) - g \int d^4y S_F(x-y)\phi(y)\Psi(y)$$

where $\psi(x)$ is the solution of a free particle Dirac equation.

Solution:
We have for $\Psi(x)$

$$\Psi(x) = \psi_i(x) - g \int d^4y S_F(x-y)\phi(x)\Psi(y)$$

where $\psi_i(x)$ is the free particle equation. Substituting the expression for $S_F(x-y)$,

$$\Psi(x) = \psi_i(x) - g(-i) \int d^4y [\theta(x_0 - y_0) \int \frac{d^3p}{(2\pi)^3} \sum_{s=1}^{2} \psi_{p,s}^+(x)\bar{\psi}_{p,s}(y)$$
$$-\theta(y_0 - x_0) \int \frac{d^3p}{(2\pi)^3} \sum_{s=3}^{4} \psi_{p,s}(x)\bar{\psi}_{p,s}(y)]\phi(y)\Psi(y)$$

Here

$$\psi_{p,s} = \frac{1}{(2p^0)^{1/2}} u(p,s) e^{-ip.x}, \;\; s = 1,2; = \frac{1}{(2p^0)^{1/2}} v(p,s) e^{ip.x}, \;\; s = 3,4$$

with $\bar{u}(p,s)u(p,s) = -\bar{v}(p,s)v(p,s) = 2m, u^{\dagger}(p,s)u(p,s) = v^{\dagger}(p,s)v(p,s) = 2p^0$

For $x^0 > y^0$ for the probablity amplitude for a final state $\psi_f(x)$ (positive energy states)

$$\int d^3x \psi_f^{\dagger}(x)\Psi(x) = \delta_{if} - ig \int d^3x \psi_f^{\dagger}(x) (\int d^4y \int \frac{d^3p}{(2\pi)^3} \sum_{s=1}^{2} \psi_{p,s}^{+}(x) \bar{\psi}_{p,s}(y)\phi(y)\Psi(y)$$

We expand $\psi_f^{\dagger}(x)$ in terms of momentum eigenstates $\psi_f^{\dagger}(x) = \int d^3q \sum_r a_{q,r}\psi_{q,r}^{\dagger}(x)$ and use $\int d^3x \sum_{q,r}(\psi_{q,r}^{\dagger}(x), \psi_{p,s}(x)) = (2\pi)^3\delta^3(q-p)\delta_{r,s}$ to obtain

$$S_{fi} = \delta_{fi} - ig \int d^4y \int d^3p \sum_s a_{p,s}\bar{\psi}_{p,s}(y)\phi(y)\Psi(y)$$

$$= \delta_{if} - ig \int d^4y \bar{\psi}_f(y)\phi(y)\Psi(y)$$

.

Identical argument leads to the same result for $y^0 > x^0$. here the final states are the negative energy states and we have an extra negative sign from the expression of S_F for $y^0 > x^0$. This leads to the factor ϵ.

34. Draw the Feynman diagramms to order g^2 for the scattering (we will call the particle represented by the field ϕ as b and by the field ψ as f)

$$b(k_i) + f(p_i) \to b(k_f) + f(p_f)$$

.

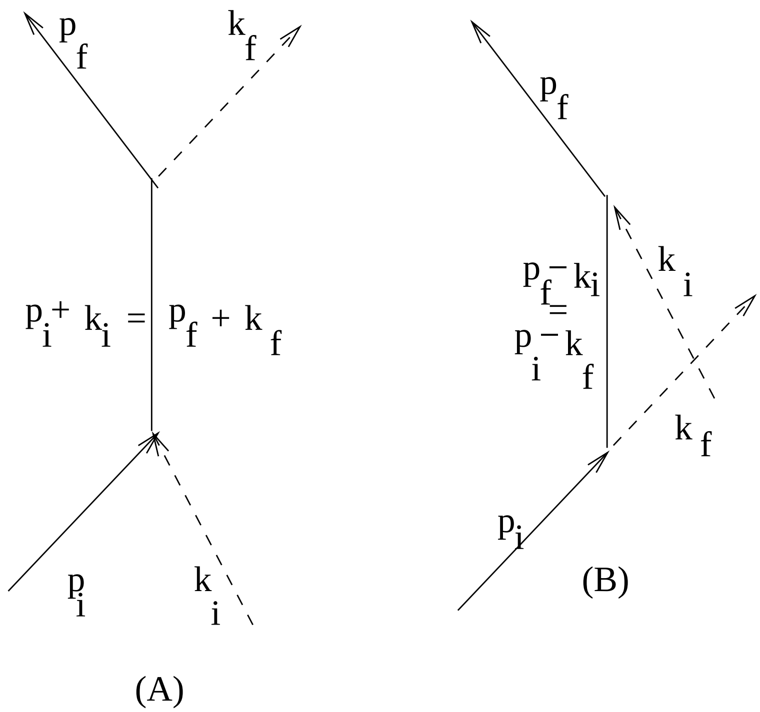
The matrix elements are

$$M_A = (-ig)^2 N (2\pi)^4 \delta^4(p_f + k_f - p_i - k_i)\bar{u}(p_f)\frac{i}{\not{p}_i + \not{k}_i - m}u(p_i)$$

and

$$M_B = (-ig)^2 N (2\pi)^4 \delta^4(p_f + k_f - p_i - k_i)\bar{u}(p_f)\frac{i}{\not{p}_i - \not{k}_f - m}u(p_i)$$

where $N = (16p_i^0 p_f^0 k_i^0 k_f^0)^{-1/2}$.



35. Let the scalar field $\phi(x)$ represent a π^- meson. Introduce the electromagnetic interaction using the gauge principle and write down the Klein Gordon eqaution in the presence of an electromagnetic vector potential A_μ. Use this to wrie an expression for J as defined in problem 31.

Solution:

Gauge principle states $p^\mu = i\partial^\mu \to p^\mu - eA^\mu = i\partial^\mu - A^\mu$ where e is the charge of the particle(π^--meson) and A^μ is the vector potential.This means the Klein-Gordon equation in the presence of an electromagnetic field takes the form

$$(\partial^\mu + ieA^\mu)(\partial_\mu + ieA_\mu)\phi(x) + m^2\phi(x) = 0$$

this gives

$$(\partial^\mu \partial_m u + m^2)\phi(x) = (-2ieA^\mu \partial_\mu + e^2 A^\mu A_\mu)\phi(x)$$

We have assumed Lorenz gauge $\partial^\mu A_\mu = 0$ in the above expression. Thus $J = (2iA^\mu \partial_\mu - e^2 A^\mu A_\mu)\phi$