Phy 523
PARTICLE PHYSICS
SOLUTION -PROBLEM SHEET VI

26. Show that the free particle Green's function is given by

$$G_0(\vec{x}', t'; \vec{x}, t) = -i \left(\frac{m}{2\pi i(t'-t)} \right) e^{\frac{im|\vec{x}'-\vec{x}|^2}{2(t'-t)^2}} \theta(t'-t)$$

Solution:
G_0 obeys the equation

$$[i\frac{\partial}{\partial t'} + \frac{\nabla'^2}{2m}]G_0(\vec{x}', t'; \vec{x}, t) = \delta^4(x'-x)$$

we write the fourier transform

$$G_0(\vec{x}'t'; \vec{x}, t) = \frac{1}{(2\pi)^4} \int d^3p d\omega e^{-i\omega(t'-t)+i\vec{p}.(\vec{x}'-\vec{x})} G(\vec{p}, \omega)$$

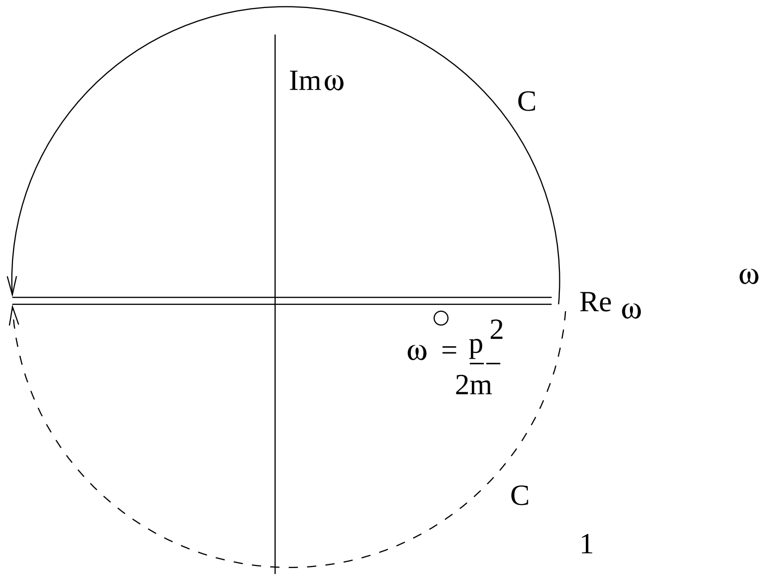
This gives on substitution (and using the fourier transform of δ-function),

$$(\omega - \frac{p^2}{2m})G_0(\vec{p}, \omega) = 1$$

or

$$G_0(\vec{x}', t'; \vec{x}, t) = \frac{1}{(2\pi)^4} \int d^3p e^{i\vec{p}.(\vec{x}'-\vec{x})} \int_{\infty}^{\infty} d\omega \frac{e^{-i\omega(t'-t)}}{\omega - p^2/2m}$$

The pole at $\omega = p^2/2m$ is fixed by the boundary condition that $G_0(\vec{x}', t'; \vec{x}, t) = 0$ when $t' < t$ As the exponential factor $e^{-i\omega(t'-t)}$tends to 0 in the upper half complex ω-plane for $t' < t$ (see Fig). Here the pole (represented by a small circle) is just below the real axis, so that the integral over the closed contour C shown is zero. For $t'-t > 0$, the contour C_1 is used (contribution from the circular part is zero) and the integral is just the $-i2\pi$ residue of the pole as the contour is traversed in the clockwise direction.



This gives

$$G_0(\vec{x}', t'; \vec{x}, t) = \frac{-i}{(2\pi)^3} \int d^3p e^{-i\frac{p^2(t'-t)}{2m} + i\vec{p}.(\vec{x}'-\vec{x})} \theta(t'-t)$$

the factor in the exponent is

$$\frac{-i(t'-t)}{2m}[p^2 - \frac{2m\vec{p}.(\vec{x}'-\vec{x})}{t'-t}]$$

$$= \frac{-i(t'-t)}{2m}[(\vec{p} - \frac{m(\vec{x}'-\vec{x})}{t'-t})^2 - \frac{m^2|\vec{x}'-\vec{x}|^2}{(t'-t)^2}]$$

Substituting this we get

$$G_0(\vec{x}', t'; \vec{x}, t) = \vec{-}i(2\pi)^3 \int d^3p e^{\frac{-i(t'-t)}{2m}(\vec{p}-\vec{a})^2} e^{\frac{im|\vec{x}'-\vec{x}|^2}{2(t'-t)}}$$

where $a = m(\vec{x}' - \vec{x})/(t' - t)$.

thus defining $\vec{q} = \vec{p} - \vec{a}$

$$G_0(\vec{x}', t'; \vec{x}, t) = \frac{-i}{(2\pi)^3} \left[\int d^3q e^{-i\frac{(t'-t)^2}{2m}q^2} \right] e^{\frac{im(|\vec{x}'-\vec{x}|^2}{2(t'-t)}} \theta(t'-t)$$

Let $\vec{r} = (i(t'-t)/2m)^{1/2}\vec{q}$. Then

$$G_0(\vec{x}', t'; \vec{x}, t) = -\frac{i}{(2\pi)^3} \int d^3r \left(\frac{2m}{i(t'-t)} \right)^{3/2} e^{-r^2} e^{\frac{im|vecx'-\vec{x}|^2}{2(t'-t)}} \theta(t'-t)$$

$$= -\frac{i}{(2\pi)^3} \pi^3 (\frac{2m}{i(t'-t)})^{3/2} e^{\frac{im|\vec{x}'-\vec{x}|^2}{2(t'-t)}} \theta(t'-t)$$

$$= -i \left(\frac{m}{2\pi i(t'-t)} \right)^{3/2} e^{\frac{im|\vec{x}'-\vec{x}|^2}{2(t'-t)}} \theta(t'-t)$$

27 Show that

$$\frac{\partial}{\partial t'} [\int d^3x' G^*(\vec{x}', t'; \vec{x}, t) G(\vec{x}', t'; \vec{y}, t)] = 0$$

and hence show

$$\int d^3x' G^*(\vec{x}', t'; \vec{x}, t) G(\vec{x}', t'; y, t) = \delta^3(\vec{x} - \vec{y})$$

Solution:

We can write the Green's function as

$$G(\vec{x}', t'; \vec{y}, t) = -i \sum_n \Phi_n(\vec{x}', t') \Phi_n^*(\vec{y}, t) \theta(t'-t)$$

where $H\Phi_n = E_n\Phi_n$. Complex conjugating and replacing $\vec{y}$ by $\vec{x}$ we get

$$G^*(\vec{x}', t'; \vec{x}, t) = i \sum_m \Phi^*(\vec{x}', t') \Phi_n(\vec{x}, t) \theta(t'-t)$$

We get the required identity.

$$\int d^3x' G^*(\vec{x}', t'; \vec{x}, t) G(\vec{x}', t'; \vec{y}, t) = \sum_n \sum_m \int d^3x' \Phi_m^*(\vec{x}', t') \Phi_n(\vec{x}', t') \Phi_m(\vec{x}, t) \Phi_n^*(\vec{y}, t)$$

$$= \sum_n \sum_n \delta_{nm} \Phi_m(\vec{x}, t) \Phi_n^*(\vec{y}, t) = \sum_n \Phi_n(\vec{x}, t) \Phi_n^*(\vec{y}, t) = \delta(\vec{x} - \vec{y})$$

using the completeness relation.

28 Use the expression from the previous problem to show

$$i\int d^3x\phi^*(\vec{x}',t')G_0(\vec{x}',t';\vec{x},t) = \phi^*(\vec{x},t)\theta(t'-t)$$

Solution:
We have for $t' > t$

$$\phi(\vec{x}',t') = i\int d^xG_0(\vec{x}',t';\vec{x},t)\phi(\vec{x},t)$$

Taking the complex conjgate,

$$\phi^*(\vec{x}',t') = -i\int d^3xG_0^*(\vec{x}',t';\vec{x},t)\phi^*(\vec{x},t)$$

Multiplying by $G_0(\vec{x}',t';\vec{y},t)$ and integrating over d^3x' we get

$$-i\int d^3xd^3x'G_0^*(\vec{x}',t';\vec{x},t)G_0(\vec{x}',t';\vec{y},t)\phi^*(\vec{x},t) = \int d^3x'\phi^*(\vec{x}',t')G_0(\vec{x}',t';\vec{y},t)$$

The left hand simplifies to (using the identity proved in the last problem)

$$-i\int d^3x\delta(\vec{x}-\vec{y})\phi^*(\vec{x},t) = -i\phi^*(\vec{y},t) = \int d^3x'\phi^*(\vec{x}',t')G_0(\vec{x}',t';\vec{y},t)$$

This proves the result

$$i\int d^3x'\phi^*(\vec{x}',t')G_0(\vec{x}',t';\vec{x},t) = \phi^*(\vec{x},t)$$

29. Using

$$S_{fi} = \int d^3x'd^3x\phi_f^*(x')G(x,x')\phi_i(x)$$

show that S is unitary.

Solution:

$$S_{fi} = i\int_{\substack{x_0\to-\infty \\ x_0'\to\infty}} d^3xd^3x'\phi_f^*(x')G(x',x)\phi_i(x)$$

$$\sum_f S_{jf}S_{fi} = \sum_f\int_{\substack{y_0,x_0\to-\infty \\ x_0',y_0'\to\infty}} d^3x'd^3xd^3y'd^3y\phi_j^*(y)G^*(y',y)\phi_f(y')\phi_f^*(x')G(x',x)\phi_i(x)$$

$$= \int d^3xd^3x'd^3yd^3y'\phi_j^*(y)G^*(y',y)\delta^3(\vec{y}' - \vec{x}')G(x',x)\phi_i(x)$$

$$= \int d^3xd^3x'd^3yd^3y'\phi_j^*(y)G^*(\vec{y}'y_0';y)\delta^3(\vec{y}' - \vec{x}')G(\vec{x}'x_0',x)\phi_i(x)$$

$$= \int d^3xd^3y'd^3y\phi_j^*(y)G^*(\vec{y}'y_0';y)G(\vec{y}'x_0',x)\phi_i(x)$$

Here $x_0', y_0' \to \infty$. This can be simplified

$$= \int d^3xd^3y\phi_j^*(y)\delta^3(\vec{x} - \vec{y})\phi_i(x) = \int d^3x\phi_j^*(x)\phi_i(x) = \delta_{ij}$$

proving the unitarity of the S-matrix.

30 Verify using the explicit expression for G_0 (Problem 26) that

$$G_0(x_2,x_1) = i\int d^3xG_0(x_2,x)G_0(x,x_1)$$

with $t_2 > t_> t_1$

Solution:
Let the integral

$$i\int d^3xG_0(x_2,x)G_0(x,x_1) = I$$

Then

$$I = i(-i)^2 \int \left(\frac{m}{2\pi i}\right)^3 \left(\frac{1}{(t_2-t)(t-t_1)}\right)^{3/2} e^{\frac{im|\vec{x}_2-\vec{x}|^2}{2(t_2-t)} + \frac{im|\vec{x}-\vec{x}_1|^2}{2(t-t_1)}} \theta(t_2-t)\theta(t-t_1)d^3x$$

$$= \left(\frac{m}{2\pi}\right)^3 (\frac{1}{(t_2-t)(t-t_1)})^{3/2} \int d^3x e^{\frac{im}{2}(\frac{x_2^2}{t_2-t} + \frac{x_1^2}{t-t_1}}$$

$$e^{\frac{im}{2}[x^2(\frac{1}{t_2-t}+\frac{1}{t-t_1}) - \frac{2xx_2}{t_2-t} - \frac{2xx_1}{t-t_1}]}$$

We simplify the term in the exponent of the second exponential (without the factor of $im/2$)

$$x^2\left(\frac{1}{t_2-t} + \frac{1}{t-t_1}\right) - 2x\left(\frac{x_2}{t_2-t} + \frac{x_1}{t-t1}\right) = \frac{x^2(t_2-t_1) - 2x(x_2(t-t_1) + (x_1(t_2-t)}{(t_2-t)(t-t_1)}$$

$$= \frac{t_2-t_1}{(t_2-t)(t-t_1)}\left(x^2 - \frac{2x(x_2(t-t_1)+x_1(t_2-t))}{t_2-t_1} + \frac{(x_2(t-t_1)+x_1(t_2-t))^2}{(t_2-t_1)^2}\right)$$

$$-\frac{t_2-t_1}{(t_2-t)(t-t_1)}\left(\frac{(x_2(t-t_1)+x_1(t_2-t))^2}{(t_2-t_1)^2}\right)$$

$$= \frac{t_2-t_1}{(t_2-t)(t-t_1)}\left[x - \frac{x_2(t-t_1)+x_1(t_2-t)}{t_2-t_1}\right]^2 - \frac{(x_2(t-t_1)+x_1(t_2-t))^2}{(t_2-t_1)(t_2-t)(t-t_1)}$$

Thus

$$I = \left(\frac{m}{2\pi}\right)^3 (\frac{1}{(t_2-t)(t-t_1)})^{3/2} e^{\frac{im}{2}\left(\frac{x_2^2(t-t_1)+x_1^2(t_2-t)}{(t_2-t)(t-t_1)} - \frac{(x_2(t-t_1)+x_1(t_2-t))^2}{(t_2-t_1)(t_2-t)(t-t_1)}\right)}$$

$$\int d^3x e^{\frac{im(t_2-t_1)}{2(t_2-t)(t-t_1)}\left[x-\frac{x_2(t-t_1)+x_1(t_2-t)}{t_2-t_1}\right]^2}$$

The exponent part of the exponential outside the integral is (apart from im/2)

$$\frac{x_2^2(t-t_1)+x_1^2(t_2-t)}{(t_2-t)(t-t_1)} - \frac{x_2^2(t-t_1)^2+x_1^2(t_2-t)^2+2x_2x_1(t-t_1)(t_2-t)}{(t_2-t_1)(t_2-t)(t-t_1)}$$

$$= \frac{x_2^2(t-t_1)(t_2-t)+x_1^2(t_2-t)(t_2-t_1)-x_2^2(t-t_1)^2-x_1(t_2-t)^2-2x_1x_2(t-t_1)(t_2-t)}{(t_2-t_1)(t_2-t)(t-t_1)}$$

$$= \frac{x_2^2(t-t_1)(t_2-t)+x_1^2(t_2-t)(t-t_1)-x^2(t-t_1)62-x_1^2(t_2-t)^2-2x_1x_2(t-t_1)(t_2-t)}{(t_2-t_1)(t_2-t)(t-t_1)}$$

$$= \frac{(t-t_1)(t_2-t)(x_2^2+x_1^2-2x_1x_2)}{(t_2-t_1)(t_2-t)(t-t_1)}$$

$$= \frac{(x_2-x_1)^2}{t_2-t1}$$

Thus

$$I = \left(\frac{m}{2\pi}\right)^3 (\frac{1}{(t_2-t)(t-t_1)})^{3/2} \int d^3x e^{\frac{im(t_2-t_1)}{2(t_2-t)(t-t_1)}(x-A)^2}$$

where

$$A = \frac{x_2(t-t_1)+x_1(t_2-t)}{t_2-t_1}$$

On doing the x-integration we get

$$I = \left(\frac{m}{2\pi}\right)^3 (\frac{1}{(t_2-t)(t-t_1)})^{3/2}\pi^{3/2}\left(\frac{2i(t_2-t)(t-t_1)}{m(t_2-t_1)}\right)^{3/2} = -i\left(\frac{m}{2\pi i(t_2-t_1)}\right)^{3/2} e^{\frac{im(x_2-x_1)^2}{(t_2-t_1)}}$$

$$= G_0(x_2, x_1)$$