Phy 523 PARTICLE PHYSICS
PROBLEM SHEET V- SOLUTIONS

21. Consider the state $|\vec{P}=0;\vec{n}>$ for a spin one particle with spin along a unit three vector $\vec{n}$, i.e. $\vec{S}.\vec{n}|\vec{P}=0;\vec{n}>=|\vec{P}=0;\vec{n}>$. ($\vec{S}$ is the spin operator) If the wave function for the particle is represented by the four vector $X^\mu(x)$ find the components of the vector field (Palne wave solution in the frame in which $\vec{P}=0$). If P is the parity operator what is $P|\vec{P}=0;\vec{n}>$ if the particle is (a) $J^P=1^-$ and (b)$J^P=1^+$?

Solution:
The plane wave solution is

$$X^\mu = e^{imt}\epsilon^\mu(P)$$

As $\epsilon^\mu P_\mu = 0$, (in the rest frame where $\vec{P}=0$) $\epsilon_0 = 0$. We reprent $\epsilon^i, i=1,2,3$ by a column vector. This means

$$\vec{S}.\vec{n}\vec{\epsilon} = \vec{\epsilon}$$

We choose the representation

$$(S_i)_{ij} = -i\epsilon_{ijk}, \quad i,j,k=1,2,3$$

this leads to

$$\vec{S}.\vec{n} = -i\begin{pmatrix} 0 & n_3 & -n_2 \\ -n_3 & 0 & n_1 \\ n_2 & -n_1 & 0 \end{pmatrix}$$

here $n_1^2+n_2^2+n_3^2=1, |\epsilon^1|^2+|\epsilon^2|^2+|\epsilon^3|^2=1$. We solve for ϵ_i

$$-i\begin{pmatrix} 0 & n_3 & -n_2 \\ -n_3 & 0 & n_1 \\ n_2 & -n_1 & 0 \end{pmatrix}\begin{pmatrix} \epsilon^1 \\ \epsilon^2 \\ \epsilon^2 \end{pmatrix} = \begin{pmatrix} \epsilon^1 \\ \epsilon^2 \\ \epsilon^3 \end{pmatrix}$$

and obtain

$$\epsilon^1 = \left(\frac{1-n_1^2}{2}\right)^{1/2}$$

$$\epsilon^2 = \left(\frac{1-n_1^2}{2}\right)^{1/2}\frac{in_1+n_2n_3}{n_3n_1-in_2}$$

$$\epsilon^3 = \left(\frac{1-n_1^2}{2}\right)^{1/2} \frac{n_3^2-1}{n_3 n_1 - i n_2}$$

For $J^P = 1^-$ we have

$$P|\vec{P}=0;\hat{n}>=-|\vec{P}=0;\vec{n}>$$

For $J^P = 1^+$ we have

$$P|\vec{P}=0;\hat{n}>=|\vec{P}=0;\vec{n}>$$

.

Note that under parity $X^i(x^0,x^i) \to -X^i(x^0,-x^i)$. So $\epsilon^i \to -\epsilon^i$. However changing the sign of ϵ^i does not change the eigenvalue of $\vec{S}.\vec{n}$ and hence $\vec{n} \to \vec{n}$ inder parity.

2. Consider the decay of $\Lambda^0(J^P = 1/2^+) \to p + \pi^-$ whose matrix element is given by

$$< p, \vec{P}_p; \pi, \vec{P}_\pi | M | \Lambda^0, \vec{P}_\Lambda >$$
$$= \bar{u}(P_p)(A + B\gamma_5)u(P_\Lambda)$$

. where A and B are functions of masses of the three particles.

Show that a term like $\bar{u}(P_p)(C\gamma_\nu P^\nu{}_\pi)u(P_\Lambda)$ and$\bar{u}(P_p)(D\gamma_5\gamma_\nu P_\pi^\nu)u(P_\Lambda)$ can be converted to terms of the form A and B.

$u(P_\Lambda)$ obeys $(\gamma_\nu P_\Lambda^\nu - m_\Lambda)u(P_\Lambda) = 0$ and a similar equation for the proton spinor.

Solution:

We have $P_\Lambda = P_p + P_\pi$. Thus the term $\bar{u}(P_p)(C\gamma_\nu P^\nu{}_\pi)u(P_\Lambda)$ can be written as

$$\bar{u}(P_p)C\gamma_\nu(P_\Lambda^\nu - P_p^\mu)u(P_\Lambda)$$

Using the equation of motion $\gamma_\nu P_\Lambda^\nu u(P_\Lambda) = m_\Lambda u(P_\Lambda)$ and $\bar{u}(P_p)\gamma_\nu P_p^\mu = m_p \bar{u}(P_p)$ we get

$$\bar{u}(P_p)C\gamma_\nu(P_\Lambda^\nu - P_p^\mu)u(P_\Lambda)$$

$=$

$$\bar{u}(P_p)C(m_\Lambda - m_p)u(P_\Lambda)$$
$$= C(m_\Lambda - m_p)\bar{u}(P_p)u(P_\Lambda)$$

Similarly,

$$\bar{u}(P_p)(D\gamma_5\gamma_\nu P_\pi^\nu)u(P_\Lambda)$$

$$= \bar{u}(P_p) D\gamma_5\gamma_\nu(P_\Lambda^\mu - P_p^\mu)u(P_\Lambda)$$
$$= \bar{u}(P_p) D\gamma_5(m_\Lambda + m_p)u(P_\Lambda)$$

$$= D(m_\Lambda + m_p)\bar{u}(P_p)\gamma_5 u(P_\Lambda)$$

Here we have used $\bar{u}(P_p)\gamma_5\gamma_\nu P_p^\mu = -m_p\bar{u}(P_p)\gamma_5$ Thus we have reduced the terms to the form needed.

23. Calulate the decay rate for $\Lambda^0 \to p + \pi^-$ where we sum over the final spins of the proton and average over the initial spin of Λ^0. (Use the expression given in Problem 22.)

Solution:

We work in the rest frame of Λ^0. The decay rate is given by with $M = \bar{u}(P_p, s_p)(A + B\gamma_5)u(P_\Lambda, s_\Lambda)$

$$\Gamma = \int \frac{d^3P_p d^3P_\pi}{(2\pi)^2 8m_\Lambda E_p E_\pi}\delta^4(P_\Lambda - P_p - P_\pi)\frac{1}{2}\sum_{\substack{s_\Lambda = 1,2 \\ s_p = 1,2}} |M|^2$$

$$= \int \frac{d^3P_\pi}{(2\pi)^2 8m_\Lambda E_p E_\pi}\delta(E_\Lambda - E_p - E_\pi)\frac{1}{2}\sum_{\substack{s_\Lambda = 1,2 \\ s_p = 1,2}} |M|^2$$

$$= \int \frac{|P_\pi|^2 d|P_\pi| d\Omega_\pi}{(2\pi)^2 8m_\Lambda E_p E_\pi}\delta(E_\Lambda - E_p - E_\pi)\frac{1}{2}\sum_{\substack{s_\Lambda = 1,2 \\ s_p = 1,2}} |M|^2$$

On performing the integration over $|P_\pi|$ using the δ-function and remebering the integration brings in a factor (the magnitude of the three momenta of the proton and the pion are equal)

$$\frac{\partial(E_p + E\pi)}{\partial|P_\pi|} = \frac{|P_\pi|}{E_p} + \frac{|P_\pi|}{E_\pi}$$

This leads to

$$= \int \frac{|P_\pi|^2 d\Omega_\pi}{(2\pi)^2 8m_\Lambda E_p E_\pi}\frac{1}{2}\sum_{\substack{s_\Lambda \\ s_p}} \left(\frac{|M|^2}{\frac{P_\pi(E_+E_\Lambda}{E_p E_\Lambda}}\right)$$

$$\int \frac{|P_\pi| d\Omega_\pi}{(2\pi)^2 8 m_\Lambda^2} \frac{1}{2} \sum_{\substack{s_\Lambda = 1,2 \\ s_p = 1,2}} \bar{u}(P_\Lambda, s_\Lambda)(A^* - B^* \gamma_5) u(P_p, s_p) \bar{u}(P_p, s_p)(A + B\gamma_5) u(P_\Lambda, s_\Lambda)$$

Note that

$$(\bar{u}(P_p, s_p)\gamma_5 u(P_\Lambda, s_\Lambda))^* = u^\dagger(P_\Lambda, s_\Lambda)\gamma_5\gamma_0 u(P_p, s_p) = -u^\dagger(P_\Lambda, s_\Lambda)\gamma_0\gamma_5 u(P_p, s_p) = -\bar{u}(P_\Lambda, s_\Lambda)\gamma_5 u(P_p,$$

When we sum over the spins we get

$$\Gamma = \int \frac{|P_\pi| d\Omega_\pi}{(2\pi)^2 8 m_\Lambda^2} \frac{1}{2} Tr((\not{P}_\Lambda + m_\Lambda)(A^* - B^*\gamma_5)(\not{P}_p + m_p)(A + B\gamma_5))$$

Using the trace identities, $Tr(\not{P}_\Lambda \ \not{P}_p) = -Tr(\not{P}_\Lambda\gamma_5 \ \not{P}_p\gamma_5) = 4P_\Lambda.P_p$ and $Tr(\not{P}_\Lambda\gamma_5 \ \not{P}_p) = 0$ we get

$$\Gamma = \int \frac{|P_\pi| d\Omega_\pi}{(2\pi)^2 8 m_\Lambda^2} 2(|A|^2(p_\Lambda.P_p + m_\Lambda m_p) + |B|^2(P_\Lambda.P_p - m_\Lambda m_p))$$

The angular integral gives 4π as $P_\Lambda.P_p = m_\Lambda E_p$ does not have any angular dependence.Thus

$$\Gamma = \frac{P_\pi}{4\pi}(|A|^2(E_p + m_p) + |B|^2(E_p - m_p))$$

24. Is parity conserved in this reaction (Problem 23)? If so why? If not why?

Solution:
The intrinsic parity of pion is negative

$$< \vec{P}_p; \vec{P}_\pi | PMP^{-1} | \vec{P}_\Lambda = 0 >= - < -\vec{P}_p; -\vec{P}_\pi | M | \vec{P}_\Lambda = 0 >$$

If parity is conserved

$$< \vec{P}_p; \vec{P}_\pi | PMP^{-1} | \vec{P}_\Lambda = 0 >=< \vec{P}_p; \vec{P}_\pi | M | \vec{P}_\Lambda = 0 >$$

This implies

$$-\bar{u}(E_p, -\vec{P}_p)(A + B\gamma_5) u(m_\Lambda, \vec{0}) = \bar{u}(E_p, \vec{P}_p)(A + B\gamma_5) u(m_\Lambda, \vec{0})$$

Using $\bar{u}(E_p, -\vec{P}_p) = \bar{u}(E_p, \vec{P}_p)\gamma_0$ and $u(m_\Lambda, \vec{0}) = \gamma_0 u(m_\Lambda, \vec{0})$, we have,

$$-\bar{u}(E_p, \vec{P}_p)\gamma_0(A + B\gamma_5)\gamma_0 u(m_\Lambda, \vec{0}) = \bar{u}(E_p, \vec{P}_p)(-A + B\gamma_5)u(m_\Lambda, \vec{0})$$

This should equal

$$< \vec{P}_p; \vec{P}_\pi | M | \vec{P}_\Lambda = 0 >= \bar{u}(E_p, \vec{P}_p)(A + B\gamma_5)u(m_\Lambda, \vec{0})$$

Finally we see that Parity conservation implies that $A = 0$.

25. From the momentum dependence (in terms of $\vec{P}_\pi = \vec{P}_p$) of decay rate calculated in problem 23, what can conclude about the angular momentum of the outgoing particles?

Solution: For small $|P_p|, /, E_p = m_p + |\vec{P}_p|^2/(2m_p)$ and thus $(E_p m_\Lambda + m_p m_\Lambda) \to 2m_p m_\Lambda$ as $|\vec{P}_p| \to 0$. Thus the term proportional to $|A|^2$ behaves as $|\vec{P}_p|$. We know from scattering theory this corresponds to an s-wave. The coeefient of $|B|^2$ term is $(E_p m_\Lambda - m_p m_\Lambda)$ which for low momentum behaves as $((m_p + |\vec{P}_p|^2/2m_p)m_\Lambda - m_p m_\Lambda) = |\vec{P}_p|^2/2m_p m_\Lambda$. The co-effient of $|B|^2$in Γ behaves as $\vec{P}_p|^3$ which corresponds to a p-wave ($l = 1$) emission.