Phy 523 PARTICLE PHYSICS PROBLEM SHEET III- SOLUTIONS

Define $S(x) = \bar\psi(x)\psi(x); P(x) = \bar\psi(x)\gamma_5\psi(x); V_\mu(x) = \bar\psi(x)\gamma_\mu\psi(x); A_\mu(x) = \bar\psi(x)\gamma_5\gamma_\mu\psi(x); T_{\mu\nu}(x) = \bar\psi\sigma_{\mu\nu}\psi(x)$; where $\sigma_{\mu\nu} = i(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)/2$. Here $\psi(x), \bar\psi(x)$ are anticommuting fields.

We use the notation below for the problems 11 to 13.

We have $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ where $\eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta = \eta_{\alpha\beta}$. The transformation of the field under a Lorentz transformation is given by $U(\Lambda)\psi(x)U(\Lambda)^\dagger = S(\Lambda)\psi'(x')$ and $U(\Lambda)\bar\psi(x)U(\Lambda)^\dagger = \bar\psi'(x')S^{-1}(\Lambda)$ Here $S(\Lambda)$ obeys the equation $S^{-1}(\Lambda)\gamma^\nu S(\Lambda) = \Lambda^\nu{}_\alpha\gamma^\alpha$. ($S(\Lambda)$ is 4x4 matrix in the Dirac space.

11. Discuss how the above bilinears transform under a proper Lorentz transformation.

Solution:

$$S'(x') = \bar\psi'(x')\psi'(x') = U(\Lambda)\bar\psi(x)U(\Lambda)^\dagger U(\Lambda)\psi(x)U(\Lambda)^\dagger$$
$$= \bar\psi(x)S^{-1}(\Lambda)S(\Lambda)\psi(x) = \bar\psi(x)\psi(x) = S(x)$$

. $S(x)$ transforms as a scalar.

$$P'(x') = \bar\psi'(x')\gamma_5\psi'(x') = U(\Lambda)\bar\psi(x)U(\Lambda)^\dagger\gamma_5 U(\Lambda)\psi(x)U(\Lambda)^\dagger$$

$$P'(x') = \bar\psi(x)S^{-1}\gamma_5 S\psi(x)$$
$$S^{-1}\gamma_5 S = S^{-1}(\frac{-i}{4!}\epsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta)S$$
$$= \frac{-i}{4!}\epsilon_{\mu\nu\alpha\beta}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\Lambda^\alpha{}_\pi\Lambda^\beta{}_\delta\gamma^\rho\gamma^\sigma\gamma^\pi\gamma^\delta = \frac{-i}{4!}\epsilon_{\rho\sigma\pi\delta}det(\Lambda)\gamma^\rho\gamma^\sigma\gamma^\pi\gamma^\delta$$

=det$(\Lambda)\gamma_5$.

det$(\Lambda) = 1$ for proper Lorentz transformations and thus

$P'(x')$ =det$(\Lambda)P(x) = P(x)$

$$V'^\mu(x') = \bar\psi'(x')\gamma^\mu\psi'(x') = \bar\psi(x)S^{-1}\gamma^\mu S\psi(x) = \Lambda^\mu{}_\beta\bar\psi(x)\gamma^\beta\psi(x) = \Lambda^\mu{}_\beta V^\beta(x)$$

V^μ transforms as a vector.

$$A'^\mu(x') = \bar\psi'(x')\gamma_5\gamma^\mu\psi'(x') = \bar\psi(x)S^{-1}\gamma_5\gamma^\mu S\psi(x)$$

$$S^{-1}\gamma_5\gamma^\mu S = S^{-1}\gamma_5 S S^{-1}\gamma^\mu S = det(\Lambda)\Lambda^\mu{}_\beta\gamma^\beta$$

Thus

$$A'^\mu(x') = det(\Lambda)\Lambda^\mu{}_\beta\bar{\psi}(x)\gamma_5\gamma^\beta\psi(x) = det(\Lambda)\Lambda^\mu{}_\beta A^\beta(x)$$

$$T'\mu\nu(x') = \bar{\psi}'(x')\sigma^{\mu\nu}\psi'(x') = \bar{\psi}(x)S^{-1}\sigma^{\mu\nu}S\psi(x)$$

$$S^{-1}\sigma^{\mu\nu}S = S^{-1}\frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)S = \Lambda^\mu{}_\alpha\Lambda^\mu{}_\beta\frac{i}{2}(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha$$

$$= \Lambda^\mu{}_\alpha\Lambda^\mu{}_\beta\sigma^{\alpha\beta}$$

Thus

$$T'^{\nu\mu}(x') = \bar{\psi}'(x')\sigma^{\mu\nu}\psi'(x') = \bar{\psi}(x)S^{-1}\sigma^{\mu\nu}S\psi(x)$$

$$= \Lambda^\mu{}_\beta\Lambda^\nu{}_\alpha\bar{\psi}(x)\sigma^{\beta\alpha}\psi(x)$$

$$= \Lambda^\mu{}_\beta\Lambda^\nu{}_\alpha T^{\beta\alpha}(x)$$

12. Discuss how the above quantities transform under parity.

Solution : Under parity we have $x'^0 = x^0, x'^i = -x^i$,det$(\Lambda) = -1$ and $S = \gamma^0$

$$U_P\bar{\psi}(x)U_P^\dagger = \bar{\psi}'(x') = \bar{\psi}(x)\gamma^0; U_P\psi(x)U_P^\dagger = \psi'(x') = \gamma^0\psi(x)$$

$$U_P S(x)U_P^\dagger = S'(x') = \bar{\psi}'(x')\psi'(x') = \bar{\psi}(x)\gamma^0\gamma^0\psi(x) = \bar{\psi}(x)\psi(x) = S(x)$$

$$U_P P(x)U_P^\dagger = P'(x') = \bar{\psi}'(x')\gamma_5\psi'(x') = \bar{\psi}(x)\gamma^0\gamma_5\gamma^0\psi(x) = -\bar{\psi}(x)\gamma_5\psi(x) = -P(x)$$

$$U_P V^\mu(x)U_P^\dagger = V'^\mu(x') = \bar{\psi}'(x')\gamma^\mu\psi'(x') = \bar{\psi}(x)\gamma^0\gamma^\mu\gamma^0\psi(x)$$

This means

$$V'^0(x') = V^0(x); V'^i(x') = -V^i(x)$$

$$U_P A^\mu(x)U_P^\dagger = A'^\mu(x') = \bar{\psi}'(x')\gamma_5\gamma^\mu\psi'(x') = \bar{\psi}(x)\gamma^0\gamma_5\gamma^\mu\gamma^0\psi(x)$$

Thus

$$A'^0 = \bar{\psi}'(x')\gamma_5\gamma^0\psi'(x') = \bar{\psi}(x)\gamma^0\gamma_5\gamma^0\gamma^0\psi(x) = -\bar{\psi}(x)\gamma^0\psi(x) = -A^0(x)$$

$$A'^i = \bar{\psi}'(x')\gamma_5\gamma^i\psi'(x') = \bar{\psi}(x)\gamma^0\gamma_5\gamma^i\gamma^0\psi(x) = \bar{\psi}(x)\gamma_5\gamma^i\psi(x) = A^i(x)$$

$$U_P T^{\mu\nu}(x)U_P^\dagger = T'^{\mu\nu}(x') = \bar{\psi}'(x')\sigma^{\mu\nu}\psi'(x') = \bar{\psi}(x)\gamma^0\sigma^{\mu\nu}\gamma^0\psi(x)$$

Thus

$$T'^{0i}(x') = \bar{\psi}(x)\gamma^0\sigma^{0i}\gamma^0\psi(x) = -T^{0i}(x)$$
$$T'^{ij}(x') = \bar{\psi}(x)\gamma^0\sigma^{ij}\gamma^0\psi(x) = \bar{\psi}(x)\sigma^{ij}\psi(x) = Tij(x)$$

13. Discuss how the above quantities transform under time revesal.

Solution:

Time reversal is an antilinear transformation. It is defined as ($x'^0 = -x^0; x^i = x^i$)

$$U_T K\psi(x)KU_T^\dagger = U_T\psi^*(x)U_T^\dagger = \psi'(x') = s\psi(x)$$

$$U_T K\bar{\psi}(x)KU_T^\dagger = U_T\bar{\psi}^*(x)U_T^\dagger = \bar{\psi}'(x') = \bar{\psi}(x)s^{-1}$$

where $K$is the complex conjugation operator and $s = \gamma^1\gamma^3; s^{-1} = -s$ and satisfies the identity $s^{-1}(\gamma^0)^*s = \gamma^0; s^{-1}(\gamma^i)^*s = -\gamma^i;$. Further $s^{-1}\gamma_5 s = \gamma_5$

$$S'(x') = U_T K\bar{\psi}(x)\psi(x)KU_T^\dagger = U_T(\bar{\psi}(x)\psi(x))^*U_T^\dagger = U_T\bar{\psi}^*(x)\psi^*(x)U_T^\dagger$$

$$= \bar{\psi}(x)s^{-1}s\psi(x) = \bar{\psi}(x)\psi(x) = S(x)$$
$$P'(x') = U_T K\bar{\psi}(x)\gamma_5\psi(x)KU_T^\dagger = U_T(\bar{\psi}(x)\gamma_5\psi(x))^*U_T^\dagger$$
$$= U_T\bar{\psi}^*(x)\gamma_5^*\psi^*(x) = \bar{\psi}(x)s^{-1}\gamma_5^*s\psi(x)$$
$$= \bar{\psi}(x)\gamma_5\psi(x) = P(x)$$

Note our definition of $P(x) = \bar{psi}(x)\gamma_5\psi(x)$ satisfies $P^\dagger = -P$ If we define a hermitian quantity $\tilde{P} = iP$ then $\tilde{P}$ is Hemitian. The transformation of $\tilde{P}$ is given by

$$\tilde{P}'(x') = U_T K i\bar{\psi}(x)\gamma_5\psi(x)KU_T^\dagger = -i\bar{\psi}(x)\gamma_5\psi(x) = -\tilde{P}(x)$$

$$V'^\mu(x') = U_T K\bar{\psi}(x)\gamma^\mu\psi(x)KU_T^\dagger = U_T(\bar{\psi}(x)\gamma^\mu\psi(x))^*U_T^\dagger$$
$$= U_T\bar{\psi}^*(x)(\gamma^\mu)^*\psi^*(x)U_T^\dagger = \bar{\psi}(x)s^{-1}(\gamma^\mu)^*s\psi(x)$$

Thus for $\mu = 0$ we have

$$V'^0(x') = \bar{\psi}(x)s^{-1}(\gamma^0)^*s\psi(x) = \bar{\psi}(x)\gamma^0\psi(x) = V^0(x)$$

and for $\mu = i$ we have

$$V'^i(x') = \bar{\psi}(x)s^{-1}(\gamma^i)^*s\psi(x) = -\bar{\psi}(x)\gamma^i\psi(x) = -V^i(x)$$

$$A'^{\mu}(x') = U_T K \bar{\psi}(x)\gamma_5\gamma^{\mu}\psi(x)KU_T^{\dagger} = U_T(\bar{\psi}(x)\gamma_5\gamma^{\mu}\psi(x))^* U_T^{\dagger}$$

$$= \bar{\psi}(x)s^{-1}(\gamma_5\gamma^{\mu})^* s\psi(x) = \bar{\psi}(x)s^{-1}\gamma_5^*(\gamma^{\mu})^* s\psi(x)$$
$$= \bar{\psi}(x)\gamma_5 s^{-1}(\gamma^{\mu})^* s\psi(x)$$

For $\mu = 0$

$$A'^0(x') = \bar{\psi}(x)\gamma_5 s^{-1}\gamma^0 s\psi(x) = \bar{\psi}(x)\gamma_5\gamma^0\psi(x) = A^0$$

For $\mu = i$

$$A'^i(x') = \bar{\psi}(x)s^{-1}\gamma_5^*(\gamma^i)^* s\psi(x) = \bar{\psi}(x)\gamma_5 s^{-1}(\gamma^i)^* s\psi(x)$$

$$= -\bar{\psi}(x)\gamma_5\gamma^i\psi(x) = -A^i(x)$$

Finally

$$T'^{\mu\nu}(x') = U_T K \bar{\psi}(x)\sigma^{\mu\nu}\psi(x)KU_T^{\dagger} = U_T\bar{\psi}^*(x)(\sigma^{\mu\nu})^*\psi(x)^* U_T^{\dagger}$$

$$= \bar{\psi}(x)s^{-1}(\sigma^{\mu\nu})^* s\psi(x)$$

For $\mu = 0, \nu = i$ we have

$$s^{-1}(\frac{i}{2})(\gamma^0\gamma^i - \gamma^i\gamma^0)^* s = s^{-1}(\frac{-i}{2})((\gamma^0)^*(\gamma^i)^* - (\gamma^i)^*(\gamma^0)^*)s$$

$$= \frac{i}{2}(\gamma^0\gamma^i - \gamma^i\gamma^0) = \sigma^{0i}$$

Thus

$$T'^{0i}(x') = T^{0i}(x)$$

For $\mu = i, \nu = j$ we have

$$s^{-1}(\sigma^{ij})^* s = -\sigma^{ij}$$

which gives

$$T'^{ij}(x') = -T^{ij}(x)$$

14. Discuss how the above quantities transform under charge conjugation.

Solution:

Under charge conjugation the transformation of the spinors fields are given by

$$\psi^c(x) = c\bar{\psi}^T(x); \bar{\psi}^c(x) = -\psi^T(x)c^{-1}$$

where the transpose is in the Dirac space and c is defined as

$$c^{-1}\gamma^\mu c = -(\gamma^\mu)^T; c = -c^T = \gamma^0\gamma^2$$

Further $c^{-1}\gamma_5 c = \gamma_5$.

$$S^c(x) = \bar{\psi}^c(x)\psi^c(x) = -\psi^T(x)c^{-1}c\bar{\psi}^T(x) = -\psi^T(x)\bar{\psi}^T(x) = -\sum_{r=1}^{4}\psi_r(x)\bar{\psi}_r(x)$$

$$= \sum_{r=1}^{4}\bar{\psi}_r(x)\psi_r(x) = \bar{\psi}(x)\psi(x)$$

Note that the transpose operator is in the Dirac space but the order of the two fermion fields have been interchanged. This gives an extra negaitve sign.

$$P^c(x) = \bar{\psi}^c(x)\gamma_5\psi^c(x) = -\psi^T(x)c^{-1}\gamma_5 c\bar{\psi}^T(x)$$

$$= -\psi^T(x)\gamma_5\bar{\psi}^T(x) = \bar{\psi}(x)\gamma_5^T\psi(x) = \bar{\psi}(x)\gamma_5\psi(x) = P(x)$$

$$(V^\mu)^c(x) = \bar{\psi}^c(x)\gamma^\mu\psi^c(x) = -\psi^T(x)c^{-1}\gamma^\mu c\bar{\psi}^T(x) = \psi^T(x)(\gamma^\mu)^T\bar{\psi}^T(x)$$

$$= -\bar{\psi}(x)\gamma^\mu\psi(x) = -V^\mu(x)$$

$$(A^\mu)^c(x) = \bar{\psi}^c(x)\gamma_5\gamma^\mu\psi^c(x) = -\psi^T(x)c^{-1}\gamma_5\gamma^\mu c\bar{\psi}^T(x) = \psi^T(x)\gamma_5(\gamma^\mu)^T\bar{\psi}^T(x)$$

$$= -\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x) = \bar{\psi}(x)\gamma_5\gamma^\mu\psi(x) = A^\mu(x)$$

$$(T^{\mu\nu})^c(x) = \bar{\psi}^c(x)\sigma^{\mu\nu}\psi^c(x) = -\psi^T(x)c^{-1}\sigma^{\mu\nu}c\bar{\psi}^T(x)$$

$$c^{-1}\frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)c = \frac{i}{2}((\gamma^\mu)^T(\gamma^\nu)^T - (\gamma^\nu)^T(\gamma^\mu)^T) = -(\sigma^{\mu\nu})^T$$

Thus

$$(T^{\mu\nu})^c(x) = \psi^T(x)(\sigma^{\mu\nu})^T\bar{\psi}^T(x)$$

$$= -\bar{\psi}(x)\sigma^{\mu\nu}\psi(x) = -T^{\mu\nu}(x)$$

15. Consider the reaction $\pi^- + d \to n + n$. It is given that the capture of π^- by deuteron occurs in the s-state (l=0). Show that $n + n$ can not be emitted in the s-state $l = 0$ state.Assuming that parity is conserved and that it is emitted in the p-state ($l = 1$), find the parity of π^-.(deuteron has spin 1 and is in the s- state (l=0); asssume the intrinsic parity of proton and neutron is positive.)

Solution:

The spin of the deuteron is $S = 1$ and since the pion is captured from the $l = 0$ state the total angular momentum of the $\pi - d$ system is $J_i = 1$. The final two neutrons are in $J_f = 1$ state. If they are emitted in the $l = 0$-state, then the total spin of the two neutrons $S = 1$. This violates Fermi-Dirac statistics as both the orbital and the spin wave functions are even under the exchange of the two neutrons. Thus the two neutrons can not be emitted in the $l = 0$ state. For the $l = 1$ state the parity of the final state is $(-1)^l = -1$. Thus the parity of the initial sate =parity of pion$\times(-1)^0$=-1. The intrinsic parity of the pion is negative.