Phy 523

PARTICLE PHYSICS-SOLUTIONS Problem sheet 2

6. Calculate the decay rate of $A \to B + C$ in the frame in which A has mo mentum $\vec{P}_A$, using the expression

$$\Gamma_{A(\vec{P}_A)\to B+C} = \int\int \frac{d^3P_B d^3P_C}{(2\pi)^2} \frac{\delta^4(P_A - P_B - P_C)}{2P_A^0 2P_B^0 2P_C^0} |M(P_A, P_B, P_C)|^2$$

and show that the result is related to the decay rate in the rest frame by

$$\Gamma_{A(\vec{P}_A)\to B+C} = \frac{m_A}{E_A} \Gamma_{A(rest)\to B+C}$$

Solution:

$$\Gamma(A(\vec{P}_A) \to B + C) = \int \frac{d^3P_B d^3P_c}{(2\pi)^2 8P_A^0 P_B^0 P_C^0} \delta^4 P_A - P_B - P_C)|M|^2$$

where $|M|^2$ is a Lorentz scalar.

$$\Gamma(A(\vec{P}_A) \to B + C) = \int \frac{d^3P_B}{(2\pi)^2 8P_A^0 P_B^0 P_C^0} \delta(E_A^0 - E_B^0 - E_C^0)|M|^2$$

where $E_B = (m_B^2 + |\vec{P}_B|^2)^{1/2}; E_C = (m_C^2 + |\vec{P}_A - \vec{P}_B|^2)^{1/2}$

$$\Gamma(A(\vec{P}_A) \to B + C) = \int \frac{d\Omega_B |\vec{P}_B|^2 d|\vec{P}_B|}{(2\pi)^2 8P_A^0 P_B^0 P_C^0} \delta(E_A^0 - E_B^0 - E_C^0)|M|^2$$

where the angular integration $d\Omega_B = d\Phi_B dcos(\theta_B)$ performing the $dcos(\theta_B)$ integration using $E_C = (m_C^2 + |\vec{P}_A|^3 + |\vec{P}_B|^2 - 2|\vec{P}_A||\vec{P}_B|cos(\theta_B))^{1/2}$ one gets

$$\Gamma(A(\vec{P}_A) \to B + C) = \int \frac{d\Phi_B |\vec{P}|^2 d|\vec{P}|}{(2\pi)^2 8P_A^0 P_B^0 P_C^0} \frac{1}{\left(\frac{|\vec{P}_A||\vec{P}_B|}{P_C^0}\right)} |M|^2$$

$$= \int \frac{d\Phi_B |\vec{P}_B| d|\vec{P}_B|}{(2\pi)^2 8P_A^0 P_B^0 |\vec{P}_A|} |M|^2$$

$$= \int \frac{d\Phi_B E_B dE_B}{(2\pi)^2 E_A E_B |\vec{P}|} |M|^2$$

where we use the identity $|\vec{P}_B||d\vec{P}_B| = E_B dE_B$.

$$\Gamma(A(\vec{P}_A) \to B + C) = \int \frac{d\Phi_B dE_B}{(2\pi)^2 8 E_A |\vec{P}_A|} |M|^2$$

$E_B$ is related to the energy in the rest frame of $A$ by the equation

$$E_B = \frac{E_A}{m_A}(E_B(r) + \frac{|\vec{P}_A||\vec{P}_B(r)|cos\theta_r|}{E_A})$$

Where$E_B(r), \vec{P}_B(r)$ and $cos\theta_r$ refer to the quantities in the rest frame of $A$. The above equation implies

$$\frac{dE_B}{\partial cos(\theta_r} = \frac{\vec{P}_A||\vec{P}_B(r)|}{m_A}$$

and

$$\Phi_B = \Phi_B(r)$$

where $\Phi(r)$ refers to the azimuthal angle in the rest frame. Thus

$$\Gamma_{(A(\vec{P}_A) \to B+C)} = \frac{1}{(2\pi)^2} \int \frac{d\Phi(r) dcos(\theta_r)|\vec{P}_A||\vec{P}_B(r)|}{8 E_A P_A m_A} |M|^2$$

$$= \frac{1}{(2\pi)^2} \int \frac{d\Omega(r)|\vec{P}_B(r)|}{E_A m_A} |M|^2$$

$$= \frac{m_A}{E_A} \Gamma_{(A(rest) \to B+C)}$$

which is the required result.

The same result can be obtained by using the identity

$$\int \frac{d^3P}{2P^0} = \int \delta(P^2 - m^2)\theta(P^0)$$

is a Lorentz scalar. Thus the integral

$$\frac{1}{2E_A} \int \frac{d^3P_C d^3P_B}{2P_B^0 2P_B^0} |M|^2 = \frac{1}{2E_A} \int d^4P_B d^4P_C \delta(P_B^2 - m_B^2)\delta(P_C^2 - m_C^2)|M|^2$$

is $\frac{1}{E_A}\times$ Lorentz scalar$=\frac{m_A}{E_A}\Gamma_{(A(rest)\to B+C)}$.

7. Show that

$$(a)\quad \gamma^\mu \not{p}\gamma_\mu = -2\not{p}$$

$$(b)\quad \gamma^\mu \not{p}\not{q}\gamma_\mu = 4p.q$$

$$(c)\quad \gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu = -2\not{c}\not{b}\not{a}$$

Solution:

(a) We have $\gamma^\nu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$ Thus

$$\gamma^\mu \not{p}\gamma_\mu = \gamma^\mu\gamma^\beta p_\beta\gamma_\mu = (-\gamma^\beta\gamma^\mu + 2\eta^{\mu\beta})p_\beta\gamma_\mu$$

$$= -\not{p}\gamma^\mu\gamma_\mu + 2p^\mu\gamma_\mu = -4\not{p} + 2\not{p} = 2\not{p}$$

(b)We use the identity $\gamma^\mu\not{p} = -\not{p}\gamma^\mu + 2p^\mu$to write

$$\gamma^\mu \not{p}\not{q}\gamma_\mu = -\not{p}\gamma^\mu\not{q}\gamma_m u + 2\not{p}\not{q} = 2\not{q}\not{p} + 2\not{q}\not{p} = 2(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)p_\mu q_\nu$$

$$= 4\eta^{\mu\nu}p_\mu q_\nu = 4p.q$$

(c) Using $\gamma^\mu\not{a} = -\not{a}\gamma^\mu + 2a^\mu$,

$$\gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu = -\not{a}\gamma^\mu\not{b}\not{c}\gamma_\mu + 2a^\mu\not{b}\not{c}\gamma_\mu = -4b.c\not{a} + 2\not{b}\not{c}\not{a}$$

$$= -4b.c\not{a} + 2(-\not{c}\not{b}\not{a} + 2b.c\not{a}) = -2\not{c}\not{b}\not{a}$$

8. Show that

$$(a)\quad Tr(\not{a}\not{b}) = 4a.b$$

(b)

$$Tr(\not{a}) = Tr(\not{a}_1\not{a}_2...\not{a}_{2n+1}) = 0 \quad n = 0, 1, ...$$

(c)

$$Tr(\gamma_5) = Tr(\gamma_5\gamma^\mu) = Tr(\gamma_5\gamma^\mu\gamma^\nu) = Tr(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho) = 0$$

(d)

$$Tr(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\epsilon^{\mu\nu\rho\sigma}$$

where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\epsilon^{0123} = 1$ and it is antisymmetric in all its indices.

Solution:

$$Tr(\gamma^\mu\gamma^\nu) = \frac{1}{2}Tr(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)) = Tr(\eta^{\mu\nu}) = 4\eta^{\mu\nu}$$

In the above we have used the fact the cyclic property $Tr(ABC..X) = Tr(XABC..)$ for any set of matrices $A, B, C, ....X$ For two matrices $A, B$ we have $Tr(AB) = Tr(BA)$.This implies

$$Tr(\not{a}\ \not{b}) = Tr(\gamma^\mu\gamma^\nu)a_\mu b_\nu = 4\eta^{\mu\nu}a_\mu b_\nu = 4a.b$$

(b)We use the identity $\gamma_5\ \not{a}\gamma_5 = -\ \not{a}$

$$Tr(\not{a}) = -Tr(\gamma_5\ \not{a}\gamma_5)$$

From the cyclical property this is equal to $-Tr(\gamma_5\gamma_5\ \not{a}) = -Tr(\not{a})$ Thus we have

$$Tr(\not{a}) = -Tr(\not{a}) = 0$$

Similarly for $n$ =odd

$$Tr(\not{a}_1\ \not{a}_2\ \not{a}_3\ \not{a}_4...\ \not{a}_n) = Tr(\not{a}_1\ \not{a}_2\ \not{a}_3\ \not{a}_4...\ \not{a}_n\gamma_5\gamma_5)$$

Using the cylical property we get

$$= Tr(\gamma_5\ \not{a}_1\ \not{a}_2\ \not{a}_3\ \not{a}_4...\ \not{a}_n\gamma_5)$$

$$= -Tr(\not{a}_1\gamma_5\ \not{a}_2\ \not{a}_3\ \not{a}_4...\ \not{a}_n\gamma_5)$$

$$= Tr(\not{a}_1\ \not{a}_2\gamma_5\ \not{a}_3\ \not{a}_4...\ \not{a}_n\gamma_5)$$

$$= -Tr(\not{a}_1\ \not{a}_2\ \not{a}_3\ \not{a}_4...\ \not{a}_n\gamma_5\gamma_5)$$

as we have moved $\gamma_5$ an odd number of times ($n$ is odd) Finally we get using $\gamma_5\gamma_5 = 1$,

$$= Tr(\not{a}_1\ \not{a}_2\ \not{a}_3\ \not{a}_4...\ \not{a}_n) = -Tr(\not{a}_1\ \not{a}_2\ \not{a}_3\ \not{a}_4...\ \not{a}_n) = 0$$

(c)

$$Tr(\gamma_5) = iTr\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0$$

$Tr(\gamma_5\gamma_\mu)$ is a trace over odd number of gamma- matrices and hence $Tr(\gamma_5\gamma_\mu) = 0$.Similar argument leads to $Tr(\gamma_5\gamma_\mu\gamma\nu\gamma_\beta) = 0$

If $\mu = \nu$ in $Tr(\gamma_5\gamma_\mu\gamma^\nu) = Tr(\gamma_5\gamma_\mu\gamma^\mu) = \eta^{\mu\mu}tr(\gamma_5) = 0$ ( In this equation no sum is implied for $\mu$).

If $\mu \neq \nu$ then let $\beta \neq \mu; \beta \neq \nu$ and $\beta$ be a spatial index. We have $\gamma_\beta\gamma_5\gamma^\mu\gamma^\nu\gamma_\beta) = -\gamma_5\gamma^\mu\gamma^\nu\gamma_\beta\gamma_\beta = \gamma_5\gamma^\mu\gamma^\nu$ ( no sum over $\beta$) This gives

$$Tr(\gamma_5\gamma^\mu\gamma^\nu) = \gamma_\beta\gamma_5\gamma^\mu\gamma^\nu\gamma_\beta)$$

Using the cyclic property of the trace,

$$Tr(\gamma_5\gamma^\mu\gamma^\nu) = Tr(\gamma_\beta\gamma_\beta\gamma_5\gamma^\mu\gamma^\nu) = -Tr(\gamma_5\gamma^\mu\gamma^\nu) = 0$$

(d) $Tr(\gamma_5\gamma^\mu\gamma^\phi\gamma^\rho\gamma^\nu) = 0$ if any two of the indices are equal as they can be contracted and reamining is trace of $\gamma_5$ with two $\gamma$ matrices, which we have already seen is zero. If all are different then we have

$$\gamma^\mu\gamma^\phi\gamma^\rho\gamma^\nu = -i\epsilon^{\mu\phi\rho\nu}\gamma_5$$

This gives

$$Tr(\gamma_5\gamma^\mu\gamma^\phi\gamma^\rho\gamma^\nu) = -i\epsilon^{\mu\phi\rho\nu}Tr(\gamma_5\gamma_5) = -i\epsilon^{\mu\phi\rho\nu}$$

9. Show that

$$\sum_{s=1}^{2} u(p,s)\bar{u}(p,s) = \not{p} + m$$

$$\sum_{s=1}^{2} v(p,s)\bar{v}(p,s) = \not{p} - m$$

where $u(p,s), v(p,s)$ represents the positive energy solution and negative energy solution respectively and $p, s$ repesent their momentum and spin.

Solution:

$u(p,s), v(p,s)$ are normalised as $\bar{u}(p,s)u(p,s') = -\bar{v}(p,s)v(p,s') = 2m\delta_{ss'}$. They also satisfy $\bar{u}(p,s)v(p,s') = \bar{v}(p,s)u(p,s') = 0$. Consider a general wave funcion $\psi(p,r)$ of momentum $p$ and spin $r$. We write is as a linear combination

$$\psi(p,r) = a(p,r)u(p,r) + b(p,r)v(p,r)$$

Let this wave function operate on $\sum_{s=1}^{2} u(p,s)\bar{u}(p,s)$ giving

$$\sum_{s=1}^{2} u(p,s)\bar{u}(p,s)\psi(p,r) = \sum_{s=1}^{2} u(p,s)\bar{u}(p,s)(a(p,r)u(p,r) + b(p,r)v(p,r))$$

Using the orthonormal conditions we get

$$\sum_{s=1}^{2} u(p,s)\bar{u}(p,s)\psi(p,r) = \sum_{s=1}^{2} u(p,s)(2m)(a(p,r)\delta_{rs})u(p,r) = 2ma(p,r)u(p,r)$$

The operation of $\not{p} + m$ on $\psi(p,r)$ gives

$$(\not{p} + m)\psi(p,r) = (\not{p} + m)(a(p,r)u(p,r) + b(p,r)v(p,r)) = 2ma(p,r)u(p,r)$$

as $\not{p}u(p,r) = mu(p,r)$ and $\not{p}v(p,r) = -mv(p,r)$. This shows when a general wave function with momentum $p$ operates on $\sum_{s=1}^{2} u(p,s)\bar{u}(p,s)$ it gives the same result as $(\not{p} + m)$ operating on $\psi(p,r)$ this means both th operators are equal.

$$\sum_{s=1}^{2} u(p,s)\bar{u}(p,s) = \not{p} + m$$

Exactly similarly we get

$$\sum_{s=1}^{2} v(p,s)\bar{v}(p,s)\psi(p,r) = (-2m)b(p,r)v(p,r)$$

which is what one gets from

$$(\not{p} - m)\psi(p,r) = -2mb(p,r)v(p,r)$$

proving the identity

$$\sum_{s=1}^{2} v(p,s)\bar{v}(p,s) = \not{p} - m$$

. Note both the right hand and the left hand side are $4x4$ marices. In fact we write it as

$$\sum_{s=1}^{2} u\alpha(p,s)\bar{u}_{\beta}(p,s) = (\not{p} + m)_{\alpha\beta}$$

where $\alpha, \beta$ take values 1 to 4. Similarly

$$\sum_{s=1}^{2} v_{\alpha}(p,s)\bar{v}_{\beta}(p,s) = (\not{p} - m)_{\alpha\beta}$$

.

10. Evaluate $\sum_{s=1}^{2}\sum_{s'=1}^{2}|\bar{u}(p,s')\ \not{a}u(q,s)|^2$ in terms of $p,q$ and $a$.

$$\sum_{s=1}^{2}\sum_{s'=1}^{2}|\bar{u}(p,s')\ \not{a}u(q,s)|^2=\sum_{s=1}^{2}\sum_{s'=1}^{2}\bar{u}(p,s')\ \not{a}u(q,s)\bar{u}(q,s)\ \not{a}u(p,s')$$

$$=\sum\alpha\beta\rho\pi=1^4\sum_{s=1}^{2}\sum_{s'=1}^{2}\bar{u}_\alpha(p,s')(\not{a})_{\alpha\beta}u_\beta(q,s)\bar{u}_\rho(q,s)(\not{a})_{\rho\pi}u_\pi(p,s')$$

$$=\sum\alpha\beta\rho\pi=1^4\sum_{s'=1}^{2}\bar{u}_\alpha(p,s')(\not{a})_{\alpha\beta}\sum_{s=1}^{2}u_\beta(q,s)\bar{u}_\rho(q,s)(\not{a})_{\rho\pi}u_\pi(p,s')$$

$$=\sum\alpha\beta\rho\pi=1^4\sum_{s'=1}^{2}\bar{u}_\alpha(p,s')(\not{a})_{\alpha\beta}(\not{q}+m)_{\beta\rho}(\not{a})_{\rho\pi}u_\pi(p,s')$$

$$=\sum\alpha\beta\rho\pi=1^4\sum_{s'=1}^{2}u_\pi(p,s')\bar{u}_\alpha(p,s')(\not{a})_{\alpha\beta}(\not{q}+m)_{\beta\rho}(\not{a})_{\rho\pi}$$

$$=\sum\alpha\beta\rho\pi=1^4(\not{p}+m)_{\pi\alpha}(\not{a})_{\alpha\beta}(\not{q}+m)_{\beta\rho}(\not{a})_{\rho\pi}$$

$$=Tr((\not{p}+m)\ \not{a}(\not{q}+m)\ \not{a})=4m^2Tr(\not{a}\ \not{a})+Tr(\not{p}\ \not{a}\ \not{q}\ \not{a})$$

As traces having odd number of $\gamma$- matrices do not contribute.

$$Tr(\not{a}\ \not{a}=4a.a$$

and

$$Tr(\not{p}\ \not{a}\ \not{q}\ \not{a})=2q.aTr(\not{p}\ \not{a})-Tr(\not{p}\ \not{a}\ \not{a}\ \not{q})$$

where we have used the identity $\not{q}\ \not{a}=2q.a-\ \not{a}\ \not{q}$. Further using $\not{a}\ \not{a}=a.a$ we get

$$Tr(\not{p}\ \not{a}\ \not{q}\ \not{a})=2q.aTr(\not{p}\ \not{a})-a.aTr(\not{p}\ \not{q})=8q.ap.a-4a.ap.q$$

Collecting all the terms we get

$$\sum_{s=1}^{2}\sum_{s'=1}^{2}|\bar{u}(p,s')\ \not{a}u(q,s)|^2=4a.a(m^2-q.p)+8q.ap.a$$