Phy 523 PARTICLE PHYSICS
PROBLEM SHEET X- SOLUTIONS

46. Write down the matrix element for the processes (include normalisation and momentum conservation) for

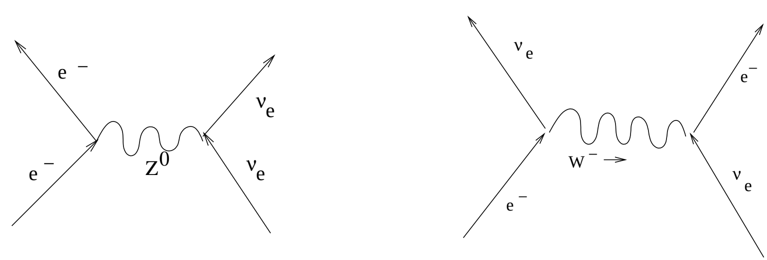
$$\nu_e + e^- \to \nu_e + e^-$$

$$\bar{\nu}_e + e^- \to \bar{\nu}_e + e^-$$

SOLUTION: Matrix element for

$$\nu_{\mathbf{e}}(\mathbf{p_1}) + \mathbf{e}^-(\mathbf{q_1}) \to \nu_{\mathbf{e}}(\mathbf{p_2}) + \mathbf{e}^-(\mathbf{q_2})$$

It occurs through an exchange of (a) W^- and (b) Z^0



Feynman diagrams for $\nu_e + e^- \to \nu_e + e^-$
The factors at the vertices are given in the parenthesis:

$$e^-\nu_e W^+ : \frac{-ig}{2\sqrt{2}}\gamma^\mu(1-\gamma_5);$$

$$\nu_e e^- W^- : \frac{-ig}{2\sqrt{2}}\gamma^\mu(1-\gamma_5)$$

$$\nu_e \nu_e Z^0 : \frac{-ig}{4cos(\theta)}\gamma^\mu(1-\gamma_5)$$

and

$$e^-e^-Z^0 : \left(\frac{ig}{4cos(\theta)}\right)\gamma^\mu((1-4sin^2(\theta))-\gamma_5)$$

The matrix elements are for (a)

$$M_a(W-exhange) = N\left(\frac{-g^2}{8}\right)\bar{u}_{\nu_e}(p_2)\gamma^\mu(1-\gamma_5)u_e(q_1)(-i)\frac{\left(\eta_{\mu\nu}-\frac{q_\mu q_\nu}{m_W^2}\right)}{(q^2-m_W^2)}$$

$$\bar{u}_e(q_2)\gamma_\mu(1-\gamma_5)u_{\nu_e}(p_1)(2\pi)^4\delta^4(q_1-q_2+p_1-p_2)$$

here $q = p_1 - q_2$ and $N = 1/(16p_1^0p_2^0q_1^0q_2^0)^{1/2}$
for(b)

$$M_b(Z-exchange) = N\left(\frac{g^2}{16cos^2(\theta)}\right)\bar{u}_e(q_2)\gamma^\alpha(A_v-A_A\gamma_5)u_e(q_1)$$

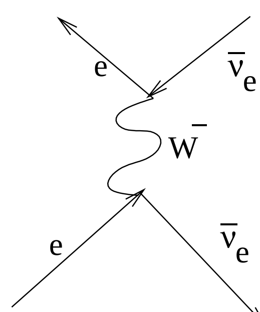
$$(-i)\frac{\left(\eta_{\alpha\beta}-\frac{r_\alpha r_\beta}{m_Z^2}\right)}{(r^2-m_Z^2)}\bar{u}_{\nu_e}(p_2)\gamma^\beta(1-\gamma_5)u_{\nu_e}(p_1)(2\pi)^4\delta^4(q_1-q_2+p_1-p_2)$$

Here

$$N = \frac{1}{(16p_1^0p_2^0q_1^0q_2^0)^{1/2}}, r = p_1-p_2, A_v = (1-4sin^2(\theta)), A_A = 1$$

.

Matrix element for the reaction $\bar{\nu}_\mathbf{e}(\mathbf{q_1}) + \mathbf{e}^-(\mathbf{p_1}) \rightarrow \bar{\nu}_\mathbf{e}(\mathbf{q_2}) + \mathbf{e}^-(\mathbf{p_2})$ goes



(a)

through an exchange of W^- and Z^0 as shown in the figure.

The matrix element is given by (the various factors for the vertices is written earlier)

$$M_a(W{-}exchange) = -N\left(\frac{-g^2}{8}\right)(2\pi)^4\delta(p_2{+}q_2{-}p_1{-}q_1)\bar{v}_{\nu_e}(q_1)\gamma^\alpha(1{-}\gamma_5)u_e(p_1)$$

$$(-i)\frac{(\eta_{\alpha\beta} - \left(\frac{r_\alpha r_\beta}{M_W^2}\right))}{r^2 - M_W^2}\bar{u}_e(p_2)\gamma^\beta(1-\gamma_5)v_{\nu_e}(q_2)$$

where $r = p_1 + q_1$ and

$$M_b(Z{-}exchange) = -N\left(\frac{-g^2}{16cos^2(\theta)}\right)(2\pi)^4\delta^4(p_2{+}q_2{-}p_1{-}q_1)\bar{u}_e(p_2)\gamma^\alpha(A_v{-}A_A\gamma_5)u_e(p_1)$$

$$(-i)\frac{\eta_{\alpha\beta} - \left(\frac{(q_1-q_2)_\alpha(p_1-q_2)_\beta}{M_Z^2}\right)}{(q_1 - q_2)^2 - M_Z^2}$$

$$\bar{v}_{\nu_e}(q_1)\gamma^\beta(1-\gamma_5)v_{\nu_e}(q_2)$$

N is the same as before. The factor $-N$ occurs rather than N is because an antiparticle (in this case an antineutrino)is present in the initial state.

47.Write down the matrix element for(include normalisation and momentum conservation)

$$e^- + u \rightarrow e^- + u$$

$$\nu_e + d \rightarrow u + e^-$$

SOLUTION: The process $\mathbf{e}^-(\mathbf{p_1}) + \mathbf{u}(\mathbf{q_1}) \rightarrow \mathbf{e}^-(\mathbf{p_2}) + \mathbf{u}(\mathbf{q_2})$ occurs through a γ and a Z^0 exchange between the electron and the u-quark. The vertex $u - u - Z^0$ is given by

$$\frac{-ig}{4cos(\theta)}(\left(1 - \frac{8sin^2(\theta)}{3}\right)\gamma^\mu - \gamma^\mu\gamma_5)$$

. $e - e - \gamma$ vertex is $ie\gamma^\mu$ and the $u - u - \gamma$ vertex is $\frac{-2ie}{3}\gamma^\mu$ where $e = gsin(\theta)$ Thus

$$M_a(\gamma{-}exchange) = \frac{2e^2}{3}N(2\pi)^4\delta^4(p_1{+}q_1{-}p_2{-}q_2)\bar{u}_e(p_2)\gamma^\mu u_e(p_1)\frac{-i}{q^2}\bar{u}_u(q_1)\gamma_\mu u_u(q_1)$$

and

$$M_b(Z^0{-}exchange) = \frac{g^2}{16cos^2(\theta)} N(2\pi)^4\delta^4(p_1{+}q_1{-}p_2{-}q_2)\bar{u}_e(p_2)(A_{ve}\gamma^\mu{-}A_{ae}\gamma^\mu\gamma_5)u_e(p_1)$$

$$(-i)\left(\frac{\eta_{\mu\nu} - \left(\frac{q_\mu q_\nu}{m_Z^2}\right)}{q^2 - m_Z^2}\right)$$

$$\bar{u}_u(q_2)(A_{vu}\gamma^\nu - A_{au}\gamma^\nu\gamma_5)u_u(q_1)$$

Here $q = p_1 - p_2, A_{ve} = 1 - 4sin^2(\theta), A_{vu} = 1 - \frac{8sin^2(\theta)}{3}, A_{av} = A_{au} = 1.$ Further$N = 1/(16p_1^0 p_2^0 q_1^0 q_2^0)^{1/2}$

The process $\nu_\mathbf{e}(\mathbf{p_1}) + \mathbf{d}(\mathbf{q_1}) \to \mathbf{e}^-(\mathbf{p_2}) + \mathbf{u}(\mathbf{q_2})$ goes through an exchange of W-meson.

$$M = N\frac{-g^2}{8}(2\pi)^4\delta^4(p_1{+}q_1{-}p_2{-}q_2)\bar{u}_e(p_2)\gamma^\mu(1{-}\gamma_5)u_{\nu_e}(p_1)(-i)\left(\frac{\eta_{\mu\nu} - \left(\frac{q_\mu q_\nu}{m_W^2}\right)}{q^2 - m_W^2}\right)$$

$$\bar{u}_u(q_2)\gamma^\nu(1 - \gamma_5)u_d(q_1)$$

Here $q = p_1 - p_2, N = (16p_1^0 p_2^0 q_1^0 q_2^0)^{1/2}$.

48. Calculate the decay rate for (a) $Z^0 \to \nu_e + \bar{\nu}_e$ (b) $Z^0 \to e^+ + e^-$.

SOLUTION: Before we consider specific decays of Z^0 or W- bosons let us calculate the decay rate for a spin -1 particle X to two fermions denoted by a and $\bar{b}$. We assume the a and $\bar{b}$ to be massless (As both Z- and W- bosons are in the range of 90-80 GeV and the leptons and quarks are very much lighter, the heaviest being a few GeV). We write the matrix element as

$$M = N\frac{g\rho}{K}(2\pi)^4\delta^4(P - q_1 - q_2)X^\mu\bar{u}_a(q_1)(A_v\gamma_\mu - A_a\gamma_\mu\gamma_5)v_b(q_2)$$

where ρ is a phase and $K = 2\sqrt{2}$ for W-decay and $K = 4cos(\theta)$ for Z-decay.P is the four momentum of X and q_1, q_2 the momenta of a and $\bar{b}$ respectively. $N = (8P^0 q_1^0 q_2^0)^{1/2}$ We consider X- to be unpolarised and so average over all the polarisation states of X. The spins of fermions is to be summed as no polarisation of spin is detected. Thus (average of initial spin is denoted by a $|\bar{M}|^2$), we get for the decay rate (the initial wave function is normalised to unity)

$$\Gamma(X \to a+\bar{b}) = \Sigma_{spins}\frac{|\bar{M}|^2}{VT} = \frac{g^2}{K^2}\int \frac{d^3q_1}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3}|N|^2(2\pi)^4\delta^4(P-q_1-q_2)\left(\frac{\eta^{-\mu\nu}+\frac{P^\mu P^\nu}{m_X^2}}{3}\right)$$

$$\sum_{s_a=1,s_b=1}^{s_a=2,s_b=2}\bar{u}_a(q_1,s_a)(A_v\gamma_\mu - A_a\gamma_\mu\gamma_5)v_b(q_2,s_b)\bar{v}_b(q_2,s_b)(A_v\gamma_\nu - A_a\gamma_\nu\gamma_5)u_a(q_1,s_a)$$

$$= \frac{g^2}{K^2}\int \frac{d^3q_1}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3}|N|^2(2\pi)^4\delta^4(P-q_1-q_2)\left(\frac{-\eta^{\mu\nu}+\frac{P^\mu P^\nu}{m_X^2}}{3}\right)$$

$$Tr((\not{q}_1)(A_v\gamma_\mu - A_a\gamma_\nu\gamma_5)(\not{q}_2)(A_v\gamma_\nu - A_a\gamma_\nu\gamma_5))$$

.

The term $\frac{P^\mu P^\nu}{m_X^2}$ does not contribute. This is seen by noting $P = q_1+q_2$ and using $\not{q}_1\ \not{q}_1 = q_1^2$. Similarly $\not{q}_2\ \not{q}_2 = 0$ This leads to (using$Tr(\not{q}_1\gamma_\mu\ /q_2\gamma^\mu\gamma_5) = 0$)

$$\Gamma = \frac{g^2}{3K^2}\int \frac{d_1^q}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3}|N|^2(2\pi)^4\delta^4(P-q_1-q_2)$$

$$[-tr(\not{q}_1(A_v\gamma_\mu - A_a\gamma_\mu\gamma_5)\ \not{q}_2(A_v\gamma^\mu - A_a\gamma^\mu\gamma_5))]$$

$$= \frac{g^2}{3K^2}\int \frac{d^3q_1d^3q_2}{(2\pi)^2 8P^0q_1^0q_2^0}\delta^4(P-q_1-q_2)$$

$$[-A_v^2Tr(\not{q}_1\gamma_\mu\ \not{q}_2\gamma^\mu) - A_a^2Tr(\not{q}_1\gamma_\mu\ \not{q}_2\gamma_5\gamma^\mu\gamma_5)]$$

We perform the trace using the identities $\gamma_\mu\ \not{q}_2\gamma^\mu = -2\ \not{q}_2$, $\gamma_\mu\gamma_5\ \not{q}_2\gamma^\mu\gamma_5 == -2\ \not{q}_2$ and $Tr(\not{q}_1\ \not{q}_2) = 8q_1.q_2, Tr(\gamma_5\ \not{q}_1\ \not{q}_2) = 0$. We also use the δ- function to integrate d^3q_2 and obtain

$$\Gamma = \frac{g^2}{3K^2}\int \frac{d^3q_1}{(2\pi)^2 8P^0q_1^0q_2^0}\delta(P^0 - q_1^0 - q_2^0)(A_v^2+A_a^2)(8q_1.q_2)$$

Squaring the four momentum relation $P = q_1+q_2$ and using the masslessness of a and $\bar{b}$ we get $m_X^2 = 2q_1.q_2$. Thus

$$\Gamma = \frac{g^2}{3K^2}\int \frac{|\vec{q}_1^2 d|\vec{q}|_1 d\Omega_1}{(2\pi)^2 8m_Xq_1^0q_2^0}\delta(m_X - q_1^0 - q_2^0)(A_v^2+A_a^2)(4m_X^2)$$

where we have assumed X- is at rest and used $P^0 = m_X$. We also have, as we are in the rest frame of X, $|\vec{q_1}| = |\vec{q_2}|$. Further, as a and $\bar{b}$ are massless, $|\vec{q_1}| = |\vec{q_2}| = q_1^0 = q_2^0$.The argument of the δ- function becomes $(m_X - 2q_1^0)$. The integration over $|\vec{q_1}|$ is the same as over q_1^0. This leads to

$$\Gamma = \frac{g^2}{3K^2}\int \frac{d\Omega_1}{(2\pi)^2 8m_X}(A_v^2 + A_a^2)(2m_X^2)$$

. We have used the identity $\int_{-\infty}^{\infty} dx\delta(ax) = |1/a|$ to get a factor of 1/2 when perfoming the integral over q_1^0. Integration over the angular variables $d\Omega_1$ gives 4π. thus we finally get

$$\Gamma = \frac{g^2}{12\pi K^2}(A_v^2 + A_a^2)m_X$$

For $\mathbf{Z^0} \rightarrow \nu_\mathbf{e} + \bar{\nu}_\mathbf{e}$, $m_X = m_Z, A_v = A_a = 1, K^2 = 16cos^2(\theta)$ giving

$$\Gamma(Z^0 \rightarrow \nu + \bar{\nu}_e) = \frac{g^2}{96\pi cos(\theta)}m_Z$$

For$\mathbf{Z^0} \rightarrow \mathbf{e}^+ + \mathbf{e}^-$ $K^2 = 16cos^2(\theta), A_v = (1 - 4sin^2(\theta)), A_a = 1$ leading to

$$\Gamma(Z^0 \rightarrow e^+ + e^-) = \frac{g^2}{192\pi cos^2(\theta)}(1(1 - 4sin^2(\theta))^2 + 1)$$

.

49. Calclate the decay rate for (a) $W^- \rightarrow e^- + \bar{\nu}_e$ (b) $W^- \rightarrow \bar{u} + d$

SOLUTION: For $\mathbf{W}^- \rightarrow \mathbf{e}^- + \bar{\mu}_\mathbf{e}$, we have $K^2 = 8, A_v = A - a = 1$. This gives

$$\Gamma(W^- \rightarrow e^- + n\bar{u}_e) = \frac{g^2}{48\pi}m_W$$

For $\mathbf{W}^- \rightarrow \mathbf{d} + \bar{\mathbf{u}}$, we have $K^2 = 8, A_v = A_a = 1$. Further we have to take into the three colours of the quarks. This gives the decay rate as

$$\Gamma(W^- \rightarrow d + \bar{u}) = \frac{g^2}{16\pi}m_W$$

.

50. Calculate the total cross section for the reaction in the c.m frame

$$\nu_e + d \rightarrow u + e^-$$

SOLUTION: We consider the reaction $\nu_{\mathbf{e}}(\mathbf{p_1}) + \mathbf{d}(\mathbf{q_1}) \to \mathbf{e}^-(\mathbf{p_2}) + \mathbf{u}(\mathbf{q_2})$

The reaction occurs through an exchange of W-boson between the leptons and the quarks. The matrix element is given by

$$M = N(2\pi)^4\delta(p_1+q_1-p_2-q_2)\left(\frac{-g^2}{8}\right)u(\bar{p_2})\gamma^\mu(1-\gamma_5)u_{\nu_e}(p_1)\left(\frac{-\eta_{\nu\mu}+\left(\frac{q_\mu q_\nu}{m_W^2}\right)}{q^2-m_W^2}\right)$$

$$\bar{u}_u(q_2)\gamma^\nu(1-\gamma_5)u_d(q_1)$$

where $q^\mu = (p_2 - p_1)^\mu$. As before the term $q^\mu q^\mu/m_W^2$ does not contribute in the massless limit of quarks and leptons.

We proceed to evaluate $|M|^2/VT$, averaging over the initial spins and summing over final spins.(we will discuss the averaging and summing over colour later)

$$\frac{|\bar{M}|^2}{VT} = \frac{|N|^2}{4}(2\pi)^4\delta^4(p_1+q_1-p_2-q_2)\frac{g^4}{64(q^2-m_W^2)^2}$$

$$\bar{u}_e(p_2)\gamma^\lambda(1-\gamma_5)u_{\nu_e}(p_1)\bar{u}_{\nu_e}(p_1)\gamma^\sigma(1-\gamma_5)u_e(p_2)$$

$$\bar{u}_u(q_2)\gamma_\lambda(1-\gamma_5)u_d(q_1)\bar{u}_d(q_1)\gamma_\sigma(1-\gamma_5)u_u(q_2)$$

$$= \frac{|N|^2}{4}(2\pi)^4\delta^4(p_1+q_1-p_2-q_2)\frac{g^4}{64(q^2-m_W^2)^2}$$

$$Tr[\not{p}_2\gamma^\lambda(1-\gamma_5)\ \not{p}_1\gamma^\sigma(1-\gamma_5)]$$

$$Tr[\not{q}_2\gamma_\lambda(1-\gamma_5)\ \not{q}_1(\gamma_\sigma(1-\gamma_5)]$$

This can be simplified using the trace relations

$$Tr[\not{p}_2\gamma^\lambda(1-\gamma_5)\ \not{p}_1(1+\gamma_5)\gamma^\sigma] = 2Tr[\not{p}_2\gamma^\lambda(1-\gamma_5)\ \not{p}_1\gamma^\sigma]$$

$$= 8(p_2^\lambda p_1^\sigma + p_2^\sigma p_1^\lambda - p_1.p_2\eta^{\lambda\sigma} + i\epsilon^{\alpha\lambda\beta\sigma}p_{2\alpha}p_{1\beta})$$

Similarly

$$Tr[\not{q}_2\gamma_\lambda(1-\gamma_5)\ \not{q}_1(1+\gamma_5)\gamma_\sigma)]$$

$$= 8(q_{2\lambda}q_{1\sigma} + q_{2\sigma}q_{1\lambda} - q_1.q_2\eta_{\lambda\sigma} + i\epsilon_{\rho\lambda\omega\sigma}q_2^\rho q_1^\omega)$$

. Thus

$$\frac{|\bar{M}|^2}{VT} = \frac{|N|^2}{4}(2\pi)^4\delta^4(p_1+q_1-p_2-q_2)\frac{g^4}{64(q^2-m_W^2)^2}64(p_2^\lambda p_1^\sigma+p_2^\sigma p_1^\lambda-\eta^{\lambda\sigma}+i\epsilon^{\alpha\lambda\beta\sigma}p_{2\alpha}p_{1\beta})$$

$$(q_{2\lambda}q_{1\sigma} + q_{2\sigma}q_{1\lambda} - q_1.q_2\eta_{\lambda\sigma} + i\epsilon_{\rho\lambda\omega\sigma}q_2^\rho q_1^\omega)$$

Using the identity $\epsilon^{\alpha\lambda\beta\sigma}\epsilon_{\rho\lambda\omega\sigma} = -2(\delta^\alpha_\rho\delta^\beta_\omega - \delta^\alpha_\omega\delta^\beta_\rho)$,and noticing the symmetry of each term with respect to λ, σ we get

$$\frac{|\bar{M}|^2}{VT} = \frac{|N|^2}{4}(2\pi)^4\delta^4(p_1 + q_1 - p_2 - q_2)\frac{g^4}{(q^2 - m_W^2)^2}4q_1.p_1q_2.p_2$$

Using the fact that flux is 2 in the centre of mass frame we get

$$\sigma = \frac{1}{2}\int\frac{d^3q_2d^3p_2}{(2\pi)^6}\frac{|N|^2}{4}(2\pi)^4\delta^4(p_1 + q_1 - p_2 - q_2)\frac{g^4}{(q^2 - m_W^2)^2}4q_1.p_1q_2.p_2$$

Integrating over d^3p_2 using the δ-function and simplifying gives

$$\sigma = \frac{g^4}{128(2\pi)^2}\int\frac{d^3q_2}{q_1^0q_2^0p_1^0p_2^0}\frac{4q_1.p_1q_2.p_2}{(q^2 - m_W^2)^2}\delta(q_1^0 + p_1^0 - q_2^0 - p_2^0)$$

we define the square of the centre of mass energy as (remebering all the leptons and quarks are considered as massless) $s = (q_1 + p_1)^2 = 2q_1.p_1 = (q_2 + p_2)^2 = 2q_2.p_2 = 4(q_1^0)^2 = 4(q_2^0)^2 = 4(p_1^0)^2 = 4(p_2^0)^2$. Further $q^2 = (q_1 - q_2)^2 = -2q_1.q_2 = -2(q_1^0)^2(1 - cos(\theta))$ where θ is the angle between $\vec{q}_1$ and $\vec{q}_2$. Substituting these and remembering the integration over δ-function gives a factor of $1/2$, we get

$$\sigma = \frac{g^4}{128(2\pi)^2}\int\frac{d\Omega_1(q_1^0)^2dq_1^0}{q_1^0q_2^0p_1^0p_2^0}\frac{s^2}{(s(1 - cos(\theta))/4 - m_W^2)^2}\delta(q_1^0 + p_1^0 - q_2^0 - p_2^0)$$

$$= \frac{g^4}{128(2\pi)^2}\int\frac{d\Omega}{2(q_1^0)^2}\frac{s^2}{(s(1 - cos(\theta))/4 - m_W^2)^2}$$

$$d\Omega = dcos(\theta)d\phi$$

We can integrate over ϕ to get a factor 2π. The integration over $cos(\theta)$ can also be performed using trignometric functions. We will take the low energy limit and assume $s << m_W^2$ when there is no dependence on θ as the factor $(s(1-cos(\theta))/4-m_W^2)^2$ becomes m_W^4 and the integration $cos(\theta)$ gives a factor 2 Putting all this together we finally have

$$\sigma = \frac{g^4s}{64\pi m_W^2}$$

.

We have not included the colour factor - in an actual scattering the quarks are not in the free state but form a part of hadron which do not carry colour. Thus we need to average over the three colours in the initial state , leading to a factor of 1/3. As the colour of each quark is preseved we have three possibilities and that gives a factor of three. Hence the colour factor is unity.