Recommended for Further Reading

Pauli Matrices

A Summer Course given at University of Hyderabad

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We summarize some important properties of Pauli Matrices.

1. The three Pauli matrices are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

2. The Pauli matrices satisfy the commutation relations.

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \tag{2}$$

3. The square of each Pauli matrix is unity. So is the square of $\hat{n}\cdot\vec{\sigma}$ where $\hat{n}$ is a unit vector.

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \hat{I}; \qquad \hat{n}\cdot\vec{\sigma}^2 = \hat{I} \tag{3}$$

4. Every Pauli matrix anticommutes with the other two Pauli matrices. There does not exist a nonzero 2×2 matrix which *anticommutes* with

   all the three Pauli matrices.

5. The above relations can be written in various different forms.

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \tag{4}$$
$$\sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk} \tag{5}$$

6. The above two relations imply that

$$\sigma_j\sigma_k = \delta_{jk} + i\epsilon_{jk\ell}\sigma_\ell \tag{6}$$

7. The above statements are can be rewritten as

   (a) $[\vec{a}\cdot\vec{\sigma},\vec{b}\cdot\vec{\sigma}] = 2i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$

   (b) $(\vec{a}\cdot\vec{\sigma})^2 = |\vec{a}|^2$

   (c) $(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) + (\vec{b}\cdot\vec{\sigma})(\vec{a}\cdot\vec{\sigma}) = 2(\vec{a}\cdot\vec{b})\hat{I}$

   (d) $(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = (\vec{a}\cdot\vec{b})\hat{I} + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$

   where $\vec{a},\vec{b}$ are two arbitrary numerical vectors.

8. The trace of each of the three matrices is zero. If we use the notation $\sigma_0 = \hat{I}$ we have the relation, we can write

$$Tr(\sigma_\mu\sigma_\nu) = 2\delta_{\mu\nu} \tag{7}$$

9. The above identity can be used to prove linear independence of Pauli matrices. The four matrices $\sigma_\mu, \mu = 0,..,3$ form a basis in the complex vector space of all 2×2 matrices.

10. Let S be complex 2×2 matrix which is expanded in terms of the matrices σ_μ

$$S = \sum_{\mu=0}^{3} C_\mu\sigma_\mu \tag{8}$$

    The expansion coefficients are given by

$$C_\mu = \frac{1}{2}Tr(S\sigma_\mu) \tag{9}$$

11. The completeness relation for the Pauli matrices is contained in the identity

$$\sum_a(\sigma^a)_{ij}(\sigma^a)_{kl} = 2\delta_{il}\delta_{jk} - \delta_{ij}\delta kl. \tag{10}$$

12. An important identity satisfied by the Pauli matrices is

$$\exp(i\vec{\alpha}\cdot\vec{\sigma}) = \cos|\vec{\alpha}| + i\vec{\alpha}\cdot\sigma\sin|\vec{\alpha}| \tag{11}$$

    where $\vec{\alpha}$ is a vector and

$$(\alpha_1,\alpha_2,\alpha_3), \qquad |\vec{\alpha}| = \sqrt{\alpha_1^2 + \alpha 2^2 + \alpha_2^2} \tag{12}$$