A BRIEF SURVEY OF THE MATHEMATICS OF QUANTUM PHYSICS

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The mathematics of quantum physics started from matrices and from differential operators. It inspired the theory of linear operators in Hilbert space and of unitary representation for symmetry groups and spectrum generating groups. The Dirac bra-ket formalism led first to Schwartz's theory of distributions and then to its generalization, the Rigged Hilbert Space (RHS) or Gelfand triplet. This Schwartz-RHS provided the mathematical justification for Dirac's continuous basis vector expansion and for the algebra of continuous observables of quantum theory. To obtain also a mathematical theory of scattering, resonance and decay phenomena one needed to make a mathematical distinction between prepared in-states and detected observables ("out-states"). This leads to a pair of Hardy RHS's and using the Paley–Wiener theorem, to solutions of the dynamical equations (Schrödinger or Heisenberg) given by time-asymmetric semi-groups, expressing Einstein causality.

Keywords: quantum physics, resonance scattering and decay, lifetime-width relation, Hardy space triplets and time asymmetric quantum dynamics.

1. Introduction

A new area of physics requires and brings about new mathematics. Often some rudimentary parts of this mathematics are already in existence, but the main development of this new mathematics is usually driven by the needs of the new physics. This was also the case for quantum physics.

2. From classical to quantum observables

Heisenberg had arrived at strange multiplication rules for position and momentum of quantum mechanics [1], of which Born immediately realized that these were rules for matrix multiplication. The collaboration of Born with Jordan [2], and the paper of Born, Heisenberg and Jordan [3] led to matrix mechanics: position, momentum and the other observables were not represented by numbers but by matrices. F. London [4] and P. Jordan [5] recognized that classical transformations

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can be carried over to transformations of an abstract linear space. Transformations of objects with continuous indices were introduced independently by Dirac [6] and Jordan [7]. The works of Born and Wiener [8] attempted to generalize the matrices to linear operators and C. Eckart [9] noticed that momentum had to be substituted by the differentiation operator with respect to position. The mathematical elaboration of the London–Jordan–Dirac transformation theory was undertaken in Göttingen by Hilbert, together with L. Nordheim and John von Neumann (1926-27) [10] and led to the Hilbert space of Lebesgue square integrable functions [11], see Section 3.1

Pauli [12] then applied this algebraic method of matrices and linear operators to the quantum mechanical Kepler problem and derived the spectrum of the hydrogen atom from the algebraic relations for momentum P_i, position Q_i, angular momentum L_i and the Runge–Lenz vector A_i, which for the Coulomb potential is an additional constant of the motion.

Shortly thereafter, Schrödinger's six papers on wave mechanics [13, 14], an attempt to return to the methods of classical wave theory for quantum physics, brought in the differential operators with continuous eigenvalues. And Dirac [15] concluded—around the time of the paper [3]—*that algebraic relations for the dynamical variables are the important properties* and introduced the first version of the Dirac formalism [16].

The outcome from this incredible period 1925/26 for physics was that quantal physical observables are mathematically described by elements of an algebra of linear operators $\mathcal{A}$, which act in a linear space Ψ in which a scalar product is defined.

3. From Dirac's formalism to the Schwartz–Rigged Hilbert space

3.1. Linear spaces with convergence

Consider such a linear space Ψ which is finite-dimensional. One can find a set of orthogonal eigenvectors $|n\rangle, n = 1, 2 \ldots, \mathcal{N}$, of an observable H, and one can expand every vector $\phi \in \Psi$ in terms of this basis system,

$$\phi = \sum_{n=1}^{\mathcal{N}} |n)(n|\phi) = \sum_{n=1}^{\mathcal{N}} |E_n)(E_n|\phi), \quad \text{where} \quad H|E_n) = E_n|E_n) \text{ and } (E_n|E_m) = \delta_{nm}, \tag{1}$$

where $\mathcal{N}$ is the dimension of Ψ.

For example, let $\Psi = \mathcal{R}^j$, the $\mathcal{N} = (2j+1)$-dimensional representation space of $SO(3)$, which is the enveloping algebra of the angular momenta J_i. If we take

$$H = \sum_{i=1}^{3} \frac{1}{2I} J_i J_i$$

(where I is the moment of inertia), then for $\phi \in \mathcal{R}^j$ with angular momentum j, the basis vector expansion is given by $\phi = \sum_{j_3=-j}^{+j} |j_3 j)(j_3 j|\phi)$, where j_3 denotes the eigenvalue of the component J_3 of the operator $\mathbf{J}$.

Using this basis vector expansion one can define the norm-square of a vector in the space Ψ,

$$||\phi||^2 \equiv (\phi|\phi) = \sum_{n=1}^{\mathcal{N}} (\phi|n)(n|\phi) = \sum_{n=1}^{\mathcal{N}} |(n|\phi)|^2 \tag{2}$$

where $|(n|\phi)|^2$ represents the probability to measure the eigenvalue E_n of the observable H in the state ϕ. One may also write this as

$$|(n|\phi)|^2 = \mathrm{Tr}(|n)(n|\phi)(\phi|) \tag{3}$$

where $|n)(n|$ is the projection operator on the energy eigenspace spanned by the $|n) \equiv |E_n)$ and where $|\phi)(\phi| \equiv \Lambda_\phi$ is the projection operator on the subspace spanned by the state vector ϕ. We shall denote any linear space in which at least one scalar product is defined by Ψ.

If Ψ is infinite-dimensional, a new problem arises because the limit $\mathcal{N} \to \infty$ needs to be defined. This means the linear space Ψ of linear combination (1) with "arbitrary large but finite $\mathcal{N}$", needs to be endowed with a meaning of *convergence* of infinite sequences. In the same linear space one can define different meanings of convergence. The original choice for the definition of convergence in the linear space Ψ was the norm convergence defined by one scalar product (ϕ, ψ). The resulting linear topological space is the Hilbert space $\mathcal{H}$ of square summable sequences: $\mathcal{H}$ is the set of all infinite linear combination $\phi = \sum_{n=1}^{\infty} |E_n\rangle c_n$ for which the infinite sum converges $\sum_{n=1}^{\infty} |c_n|^2 < \infty$; $c_n = \langle E_n|\phi\rangle$. This space $\mathcal{H}$ is David Hilbert's (realization of the) Hilbert space [17],

$$\mathcal{H} = \{\phi \ |(\phi, \phi) = \sum_{n=1}^{\infty} |(E_n|\phi)|^2 < \infty\}. \tag{4}$$

In this linear topological space an infinite sequence $\phi_1, \phi_2, \phi_2, \ldots, \phi_n, \phi_{n+1}, \ldots$ is a converging (precisely a Cauchy) sequence if and only if $||\phi_n - \phi_m|| \to 0$ for $n, m \to \infty$ and in the Hilbert space (4) for every Cauchy sequence there exists a $\phi \in \mathcal{H}$ such that $\phi : \phi_n \xrightarrow{\mathcal{H}} \phi$, which means $||\phi_n - \phi|| \to 0$ as $n \to \infty$. $\mathcal{H}$ is a *complete* linear scalar product space, since from $||\phi_n - \phi_m|| \to 0$ follows, there exists a unique ϕ in $\mathcal{H}$ such that $||\phi_n - \phi|| \to 0$ for $n \to \infty$.

However, the convergence $\phi_n \xrightarrow{\mathcal{H}} \phi$ is not the only way to define convergence in an infinite-dimensional linear space Ψ. For example, one can choose the topological space of rapidly decreasing sequences [18] $\{(E_n|\phi)\}_{n=1}^{\infty}$ which fulfill $\sum_n (n + 1/2)^{2p} |(E_n|\phi)|^2 < \infty$ which is the set of all linear combinations $(E_n \sim (n + 1/2))$:

$$\phi = \sum_{n=1}^{\infty} |E_n)(E_n|\phi), \qquad H|E_n) = E_n|E_n). \tag{5}$$

This topological space denoted by

$$\Phi = \{\phi \,|(\phi, (H+I)^{2p}\phi) = \sum_{n=1}^{\infty}(E_n+1)^{2p}|(E_n|\phi)|^2 < \infty\} \tag{6}$$

is called a countably normed space because its convergence is defined by a countable number of norms

$$||\phi||_p = \big((H+I)^p\phi, (H+I)^p\phi\big), \qquad p = 0, 1, 2, \ldots, \tag{7}$$

and thus convergence of an infinite sequence of elements ϕ_n in $\Phi : \phi_n \xrightarrow{\Phi} \phi$, means that $||\phi_n - \phi||_p \to 0$ for every $p = 0, 1, 2, \ldots$

It is not hard to see that

$$\Phi \subset \mathcal{H}. \tag{8}$$

A topological space is complete if every Cauchy sequence has a limit point in the space. The Hilbert space (4) is complete with respect to the norm topology (definition of convergence defined by the single norm (2)). The space Φ is complete with respect to the topology given by the countable number of norms (7).

The definition of *convergence* in a representation space for an algebra of observables $\mathcal{A}$ determines the algebra of observables. The definition of convergence is as important as the defining algebraic relations of $\mathcal{A}$. The norms are usually defined in terms of (some) operators of the algebra.

Using the space Φ one can see that the operators P, Q and

$$H = \frac{1}{2m}P^2 + \frac{m\omega^2}{2}Q^2$$

of the harmonic oscillator are continuous operators defined in the entire space Φ [18]. But these operators are not continuous operators with respect to the norm convergence and are not defined everywhere in the Hilbert space $\mathcal{H}$ of (4). The operators P, Q, H form an algebra of continuous operators in Φ but not in $\mathcal{H}$. That quantum mechanical observables are noncontinuous operators in $\mathcal{H}$ is one of the major mathematical difficulties with the Hilbert space, because the operators like Q (multiplication) and P (differentiation) cannot be multiplied everywhere in $\mathcal{H}$.

In general, the Hamilton operator has, in addition to the discrete spectrum $E_1, E_2, \ldots$, also continuous energy values E, $0 \leq E < \infty$. The values for momentum and position are in general also continuous $\mathbf{p}, \mathbf{x} \in \mathbb{R}^3$. Thus physics suggests that in addition to the discrete basis vector expansion (1) also eigenvectors with a continuous set of eigenvalues should exist, and that in general

$$\phi = \sum |E_n\rangle\langle E_n|\phi\rangle + \int_0^{\infty} dE|E\rangle\langle E|\phi\rangle, \tag{9}$$

where

$$H|E\rangle = E|E\rangle \qquad \text{with} \quad 0 \leq E < \infty. \tag{10}$$

For position $\vec{x}$ and momentum $\vec{p}$ in three dimensions similar expansions should exist:

$$\phi = \iiint_{-\infty}^{\infty} d^3\vec{x}\ |\vec{x}\rangle\langle\vec{x}|\phi\rangle, \qquad \phi = \iiint_{-\infty}^{\infty} d^3\vec{p}\ |\vec{p}\rangle\langle\vec{p}|\phi\rangle. \tag{11}$$

These basis vector expansions (1), (9), (11) are formal generalizations of the ordinary basis vector expansion

$$\mathbf{r} = \sum_{i=1}^{3} \mathbf{e}_i x^i \tag{12}$$

to $\mathbf{r} \in \mathbb{R}^3$ corresponds component

$$x^i = (\mathbf{e}_i \cdot \mathbf{r}) = (\mathbf{r} \cdot \mathbf{e}_i), \qquad \mathbf{r} \cdot \mathbf{v} = \sum_{i=1}^{3} x^i v^i \tag{13}$$

from the 3-dimensional space $\mathbb{R}^3$ to an infinite-dimensional space, and even to a space of continuously infinite dimensions (9) and (11). In analogy to the scalar product in $\mathbb{R}^3$ (13) the scalar product of two vectors ψ and ϕ, in Ψ should then be defined as

$$\overline{(\phi|\psi)} = (\psi|\phi) = \sum_n \langle\psi|E_n\rangle\langle E_n|\phi\rangle + \int_0^{\infty} dE\ \langle\psi|E\rangle\langle E|\phi\rangle. \tag{14}$$

For the case that there are no discrete eigenvalues, this becomes

$$\langle\psi|\phi\rangle = \int_0^{\infty} dE\langle\psi|E\rangle\langle E|\phi\rangle. \tag{15}$$

Since in (2) $(\phi|n) = \overline{(n|\phi)}$, we expect that $\langle\psi|E\rangle = \overline{\langle E|\psi\rangle}$ should hold.

The **bracket** for two vectors ψ, $\phi \in \Phi$ is the $\mathcal{H}$-space scalar product $\langle\phi|\psi\rangle = (\phi, \psi)$. The brackets that involve the kets $|E\rangle$ with continuous eigenvalues E, the $\langle\phi|E\rangle$ with $\phi \in \Phi$ are *not* scalar products but antilinear continuous functionals $F_E(\phi) = \langle\phi|E\rangle$ of ϕ for all $\phi \in \Phi$. This will be defined below in (31).

In order that (15) has a mathematical meaning (a finite value), the norm of the vectors must be finite,

$$||\phi||^2 = (\phi, \phi) = \int dE\langle\phi|E\rangle\langle E|\phi\rangle = \int dE|\langle E|\phi\rangle|^2 < \infty. \tag{16}$$

That means the wave functions $\phi(E) = \langle E|\phi\rangle$ must be square integrable (and similar for the position and momentum wave functions $\phi(x) = \langle x|\phi\rangle$ and $\phi(p) = \langle p|\phi\rangle$).

This leads to the first problem with the mathematics of quantum physics if one wants to use (the ordinary and familiar and intuitive) Riemann integrals in the definition (16). The space of Riemann square integrable functions {e.g. $\phi(E) = \langle E|\phi\rangle$}, is not *complete* with respect to the norm. This means:

From $||\phi_\nu - \phi_\mu|| \to 0$ for $\mu, \nu \to \infty$, i.e. from ϕ_ν, $\nu = 1, 2, \ldots,$ being a Cauchy ("convergent") sequence it does *not* follow:

There exists a ϕ in the linear space Ψ, such that $\phi_\nu \to \phi$, i.e. $||\phi_\nu - \phi|| \to 0$.

There are two ways to overcome this problem:

1. Stick to the Hilbert space $\mathcal{H}$, i.e. a linear topological space in which the convergence $\phi_\nu \to \phi$ is defined by one norm as $||\phi_\nu \to \phi|| \to 0$, *but* use Lebesgue integrals instead of Riemann integrals for (16), and define the scalar product as scalar product:

$$(\psi, \phi) = \int_{\text{Lebesgue}} dE(\psi|E)(E|\phi). \tag{17}$$

This definition of scalar product leads to the norm of state vectors given by the Lebesgue integral

$$||\phi||^2 = (\phi|\phi) = \int_{\text{Lebesgue}} dE|(E|\phi)|^2 < \infty, \qquad (E|\phi) \in \mathcal{L}^2. \tag{18}$$

The space of Lebesgue square integrable functions, von Neumann's Hilbert space [10],

$$\mathcal{L}^2 = \{\phi(E) : ||\phi||^2 = (\phi|\phi) = \int_{\text{Lebesgue}} dE|(E|\phi)|^2 < \infty\} \tag{19}$$

is much larger than the set of Riemann-square integrable smooth functions of (16).

Whereas the space of Riemann square integrable functions (16) is not complete, the space $\mathcal{L}^2$ is complete. It provided a mathematical framework for quantum mechanics and was an enormous mathematical accomplishment, starting with Hilbert's lectures in Göttingen 1927/28 [10] and resulting in von Neumann's book [11].

Von Neumann's Hilbert space does not provide by itself an easy theory of quantum physics for the following reasons:

(1) Most observables, including Q, P fulfilling $[Q, P] = i\hbar I$, are unbounded (non-continuous) operators in Hilbert space $\mathcal{H}$ (i.e., $\exists\, \chi \in \mathcal{H}$ such that $||Q\chi|| \to \infty$). Therefore the algebraic relations of P and Q etc. (addition and multiplication) cannot be applied to the whole Hilbert space $\mathcal{H}$, but only to their common invariant domains. Many observables are generators of noncompact groups which are represented by unbounded operators in $\mathcal{H}$. This means that the whole Hilbert space does not admit these algebras of physically important observables given by the enveloping algebras of noncompact (symmetry-) groups.

(2) One does not have a Dirac basis vector expansion like (9) or (11) because Dirac kets $|E\rangle$, $|x\rangle$, etc., do not exist in the Hilbert space $\mathcal{H}$ (the value of the function $\phi(E) = (E|\phi)$ at one point E does not make sense in (19)). In one respect the Hilbert space $\mathcal{H}$ is too big (for the algebra of observables) and on the other hand it is too small (no Dirac kets).

2. Therefore we have to go beyond the Hilbert space and follow the path shown by Dirac and *construct* the mathematics that makes Dirac's formalism into a mathematical theory:

Replace the norm-convergence of the Hilbert space by a new definition of convergence using a countable number of norms (as e.g. in (7)) such that two conditions are fulfilled

(1) Quantum observables are represented by an algebra $\mathcal{A}$ of linear continuous operators defined everywhere in a scalar product space Φ (i.e. define the topology in the linear space Ψ such that every physically observable $A \in \mathcal{A}$ fulfilling some defining (commutation) relations is represented by a continuous operator[1]).
(2) For every quantum physical system there exists a Complete Set of Commuting Observables. The CSCO furnishes a basis system of the space Φ, i.e. every $\phi \in \Phi$ can be expanded by a Dirac basis vector expansion with respect to the eigenkets of the CSCO.
The space Φ of (6) is an example of a countably normed scalar product space for which Dirac basis vector expansion (9) or (11) holds as the nuclear spectral theorem.

3.2. From Dirac kets to Schwartz distributions

We start our presentation with the Dirac formalism and then show how it can be turned into a mathematical theory.

As an example, we consider a quantum system that has a spherically symmetric Hamiltonian H, and whose CSCO is given by the operators

$$H, J^2 \text{ and } J_3$$

(where J_i are the angular momentum operators). This CSCO has the common eigenkets $|Ejj_3\rangle$:

$$H|Ejj_3\rangle = E|Ejj_3\rangle, \tag{20}$$

$$J^2|Ejj_3\rangle = \hbar^2 j(j+1)|Ejj_3\rangle, \quad J_3|Ejj_3\rangle = \hbar j_3|Ejj_3\rangle. \tag{21}$$

The eigenvalues of H may be discrete and/or continuous depending upon the particular Hamiltonian of the system that we consider. Dirac postulated that every $\phi \in \Psi$ can be expanded as

$$\phi = \sum_{E_n jj_3} |E_n jj_3)(E_n jj_3|\phi) + \sum_{jj_3} \int_0^\infty dE|Ejj_3\rangle\langle Ejj_3|\phi\rangle. \tag{22}$$

For discrete E_n, the $|E_n jj_3)$ are the usual eigenvectors fulfilling the orthogonality conditions

$$(E_{n'}j'j_3'|E_n jj_3) \equiv \big(|E_{n'}j'j_3'), |E_n jj_3)\big) = \delta_{n'n}\delta_{j'j}\delta_{j_3'j_3} = \delta_{E_{n'}E_n}\delta_{j'j}\delta_{j_3'j_3}, \tag{23}$$

where

$$\delta_{n'n} = \begin{cases} 1 & \text{for } n' = n, \\ 0 & \text{for } n' \neq n, \end{cases} \tag{24}$$

and similar $\delta_{j'j}$ $\delta_{j_3'j_3}$; they are the Kronecker deltas.

[1] An operator A is continuous if it maps every continuous sequence $\phi_n \to \phi$ into the continuous sequence $A\phi_n \to A\phi$.

For continuous E, the $|Ejj_3\rangle$ are the Dirac kets. They are not in the space Ψ or the Hilbert space $\mathcal{H} \supset \Psi$. They are new eigenvectors which fulfill the "Dirac orthogonality condition"

$$\langle E'j'j_3'|Ejj_3\rangle = \delta(E' - E)\delta_{j'j}\delta_{j_3'j_3}. \tag{25}$$

Here $\delta(E)$ is Dirac's "δ-function" introduced by Dirac [16] to fulfill the defining relation

$$\int \delta(E)dE = 1, \qquad \delta(E) = 0 \quad \text{for} \quad E \neq 0. \tag{26}$$

The most important property of $\delta(E)$ is

$$\int \delta(E - E_0)f(E)dE = f(E_0) \tag{27}$$

"where $f(E)$ is any continuous function" of E. A mathematical function with these properties of the $\delta(E)$ did not make sense in the classical mathematics of Dirac's time. But as generalized functions mathematical objects like the Dirac-δ where introduced in 1936 [19], and in 1950 Schwartz presented his theory of distributions [20], making use of earlier results from the general theory of linear topological spaces. Distributions are defined always in connection with a set of "well behaved function" as continuous antilinear functionals on a space of well-behaved functions.

The Kronecker-δ (24) fulfills the equation

$$\cdot \sum_{n'} \delta_{n'n}(E_{n'}jj_3|\phi) = (E_n jj_3|\phi) \tag{28}$$

for the space of rapidly decreasing sequences like those of (6)

$$\{(E_1|\phi), (E_2|\phi), (E_3|\phi), \ldots (E_n|\phi) \ldots\}.$$

In analogy of the property (28) the distribution $\delta(E' - E)$ is defined as the mathematical object that fulfills the identity

$$\int dE' \, \delta(E' - E)\langle E'jj_3|\phi\rangle = \langle Ejj_3|\phi\rangle \tag{29}$$

or

$$\int dE' \, \delta(E' - E)\phi(E') = \phi(E) = \langle E|\phi\rangle \tag{30}$$

for a space of *well-behaved* functions $\langle Ejj_3|\phi\rangle = \phi_{jj_3}(E) \equiv \phi(E)$. This space of *well-behaved* functions $\{\phi(E)\}$ cannot include all square integrable functions. The functions for which (30) can be fulfilled must be better, "more well-behaved" functions $\phi(E)$ than those fulfilling (19). Schwartz required that the "well-behaved" functions $\phi(E)$ be continuous and infinitely differentiable. In addition, he required that the $\phi(E)$ and all orders of their derivatives decrease faster than any inverse power of E as $E \to \infty$. These properties *define* the Schwartz function space,

$\mathcal{S} = \{\phi(E)\}$ [20] (one also has other choices for "well-behaved" function spaces).

One can generalize the Dirac-δ of (30) and consider the set of mathematical objects $F(E)$ which associate to every well-behaved function $\phi(E) \in \mathcal{S}$ a well-specified number

$$\int dE'\, F(E' - E)\phi(E') = \phi_F(E). \tag{29_F}$$

This set $\{F(E)\}$ is called the set of tempered distributions. It is larger than the set of well-behaved functions and the space of Lebesgue square integrable functions $\mathcal{L}^2$ of (19):

$$\{F(E)\} \supset \mathcal{L}^2 \supset \{\phi(E)\}.$$

Grothendieck [21], Gel'fand and his school [22], Maurin [23], Pietch [24] and others generalized Schwartz's theory of distributions. They defined abstractly (by specifying their properties) a class of linear, topological spaces in which the nuclear spectral theorem holds, and constructed the rigged Hilbert space [22, 23], as a triplet of spaces in which Dirac's basis kets expansion (22) can be proven as the nuclear spectral theorem. The space of Schwartz functions $\mathcal{S}$ and the space of its continuous antilinear functionals $\mathcal{S}^\times$ on $\mathcal{S}$, is one of the possible realizations of a rigged Hilbert space [22] (i.e. representations of the RHS by function spaces).

The Schwartz function space fulfills: $\mathcal{S} \subset \mathcal{L}^2$ (because *some* but not all of the $\mathcal{L}^2$-function classes contain a smooth function $\phi(E) \in S$). From $\mathcal{S} \subset \mathcal{L}^2$ it follows for the spaces of continuous anti-linear functionals: $\mathcal{L}^{2^\times} \subset \mathcal{S}^\times$. Then using the Fréchet–Riesz theorem [25] which states that $\mathcal{L}^{2^\times} = \mathcal{L}^2$ one obtains a triplet of function spaces

$$S \subset \mathcal{L}^2 = (\mathcal{L}^2)^\times \subset S^\times, \tag{30}$$

the rigged Hilbert space of Schwartz functions.

3.3. The rigged Hilbert space

The abstract Schwartz-rigged Hilbert space is the triplet of vector spaces (abstract linear topological spaces) which is defined in the following way: Start with the abstract (complete) Hilbert space $\mathcal{H}$ (as defined by the $\mathcal{H}$-space axioms) and consider

$$\mathcal{H}^\times = \text{ set of all continuous antilinear functionals on } \mathcal{H}.$$

This means

$$\mathcal{H}^\times \equiv \{F(\phi) | F(\alpha\phi_1 + \beta\phi_2) = \overline{\alpha}F(\phi_1) + \overline{\beta}F(\phi_2);$$
$$\text{from } \phi_\nu \xrightarrow{\mathcal{H}} \phi \text{ in } \mathcal{H} \text{ follows } F(\phi_\nu) \to F(\phi) \text{ with respect to } \mathbb{C}\}. \tag{31}$$

Then, according to the Fréchet–Riesz theorem [25] for every continuous functional $F(\phi)$, there exists a vector $\psi \in \mathcal{H}$ such that this functional is given by the scalar product

$$F(\phi) = (\phi, \psi). \tag{32}$$

This allows us to identify $F \in \mathcal{H}^{\times}$ with the vector $\psi \in \mathcal{H}$,

$$\mathcal{H}^{\times} \ni F(.) \equiv |\psi) \in \mathcal{H}, \tag{33}$$

$$\mathcal{H}^{\times} = \mathcal{H}. \tag{34}$$

Choose a subspace $\Phi \subset \mathcal{H}$ and define a stronger definition of convergence $\xrightarrow{\Phi}$ in Φ than the Hilbert space convergence $\xrightarrow{\mathcal{H}}$ (that means from $\phi_\nu \xrightarrow{\Phi} 0 \Rightarrow \phi_\nu \xrightarrow{\mathcal{H}} 0$, every $\xrightarrow{\Phi}$ convergent series is also a $\xrightarrow{\mathcal{H}}$ convergent but not vice versa). Let Φ be complete, reflexive $(\Phi^{\times})^{\times} = \Phi$, and nuclear (i.e. countably normed with Hilbert–Schmidt embeddings). Then with $\mathcal{H} = \mathcal{H}^{\times}$ one obtains a triplet of spaces

$$\Phi \subset \mathcal{H} = \mathcal{H}^{\times} \subset \Phi^{\times}. \tag{35}$$

Here

$\Phi^{\times}$ = space of all antilinear continuous functionals on Φ, i.e. it is the set of functions $F(\phi)$ on Φ which fulfill

$$F(\alpha\phi_1 + \beta\phi_2) = \overline{\alpha}F(\phi_1) + \overline{\beta}F(\phi_2) \tag{36}$$

and

$$F(\phi_\nu) \to F(\phi) \text{ for every convergent sequence } \phi_\nu \xrightarrow{\Phi} \phi \in \Phi. \tag{37}$$

Eq. (36) means that the functional F is antilinear and (37) means that F is a continuous functional on Φ. In Dirac's notation (36) is written as

$$\langle \alpha\phi_1 + \beta\phi_2 | F \rangle = \overline{\alpha}\langle \phi_1 | F \rangle + \overline{\beta}\langle \phi_2 | F \rangle. \tag{38}$$

Thus the functional $\langle \phi | F \rangle \equiv F(\phi)$ is an extension of the scalar product (ϕ, ψ) for $\psi \in \mathcal{H}^{\times}$ to the space $\Phi^{\times}$.

A generalized eigenvector of the operator A corresponding to the generalized eigenvalue E is a continuous antilinear functional $F \in \Phi^{\times}$, such that

$$\langle A\phi | F \rangle = \langle \phi | A^{\times} F \rangle = E \langle \phi | F \rangle \qquad \text{for every } \phi \in \Phi. \tag{39}$$

Following Dirac's notation and denoting the eigenfunctional F of (39) by the Dirac ket $|E\rangle : F = |E\rangle \in \Phi^{\times}$, one writes (39)

$$\langle A\phi | E \rangle = \langle \phi | A^{\times} | E \rangle = E \langle \phi | E \rangle \qquad \text{for all } \phi \in \Phi \tag{40}$$

or one writes simply

$$A^{\times} |E\rangle = E|E\rangle, \qquad |E\rangle \in \Phi^{\times}. \tag{41}$$

Dirac's basis vector expansion (9) is the *Nuclear Spectral Theorem* [22]. This mathematical theorem states:

For every (essentially)[2] self-adjoint operator H in Φ there exists a complete set of $|E\rangle \in \Phi^\times$ such that

$$\phi = \int dE|E\rangle\langle E|\phi\rangle \qquad \text{for every } \phi \in \Phi. \tag{42}$$

Iff Φ is the abstract Schwartz space then the continuous components $\langle E|\phi\rangle$ of the vector ϕ "along" the basis vector $|E\rangle$ are the Schwartz functions

$$\{\langle E|\phi\rangle\} = \{\phi(E)\} = \mathcal{S} \Leftrightarrow \Phi = \{\phi\}, \tag{43}$$

and the bra-ket of two generalized eigenkets is the Dirac-δ "function"

$$\langle E'|\phi\rangle = \int dE\langle E'|E\rangle\langle E|\phi\rangle, \qquad \langle E'|E\rangle = \delta(E' - E) \in \mathcal{S}^\times. \tag{44}$$

S is called a realization of the abstract Schwartz space Φ and $S^\times$ (space of tempered distributions) is called a realization of the space $\Phi^\times$ of continuous (with respect to τ_Φ) functionals on Φ.

3.4. A Schwartz rigged Hilbert space for the Dirac formalism of quantum mechanics

The original motivation for the introduction of the RHS in mathematics was the general theory of eigenfunction expansion of differential operators [22, 23]. The original motivation in (1964–1966) for introducing the RHS in quantum mechanics [26–29] was to provide a mathematical theory for the Dirac formalism [3033].

The rigged Hilbert space triplet for a particular quantum system (e.g. oscillator, rotator, Kepler system, etc.) is determined by its algebra of observables. One constructs a RHS

$$\Phi \subset \mathcal{H} \subset \Phi^\times \tag{45}$$

such that the algebra of observables is represented by an algebra of *continuous* operators $\mathcal{A}$ in a Schwartz space Φ. The conjugate operators $A^\times \in \mathcal{A}^\times$ in the dual space of Φ are defined by using (39), i.e. by

$$\langle\phi|A^\times F\rangle = \langle A\phi|F\rangle \qquad \text{for all } \phi \in \Phi, F \in \Phi^\times.$$

$\mathcal{A}^\times$ also forms an algebra of operators, and they are continuous operators in the space of continuous functionals $\Phi^\times$. The nuclear spectral theorem then assures that

[2] H here denotes the operator in Φ continuous with respect to the definition of convergence τ_Φ in Φ. It is not a closed operator in the space $\mathcal{H}$. The closure of H in the completion of Φ with respect to the Hilbert space convergence (norm-convergence $\tau_{\mathcal{H}}$) is usually denoted by $\overline{H}$. The operator H is called essentially self-adjoint if $\overline{H} = H^\dagger$. Let A be a continuous operator in Φ (with respect to τ_Φ) then $A^\times$ is defined by

$$\langle A\phi|F\rangle = \langle\phi|A^\times|F\rangle \qquad \text{for all} \qquad \phi \in \Phi,\ F \in \Phi^\times,$$

is a continuous operator in $\Phi^\times$. If A is essentially self-adjoint in $\mathcal{H}$ then we have a triplet of operators corresponding to the triplet of spaces (35),

$$A \subset \overline{A} = A^\dagger \subset A^\times.$$

for a *complete set of commuting observables*

$$A_1, A_2, \dots, A_N, \qquad [A_k, A_\ell] = 0, \tag{46}$$

there is a system of generalized eigenvectors

$$A_i|\lambda_1, \lambda_2, \dots, \lambda_N\rangle = \lambda_i\, |\lambda_1, \lambda_2, \dots, \lambda_N\rangle \tag{47}$$

such that every $\phi \in \Phi$ can be expanded with respect to this basis system

$$\phi = \int d\lambda_1 \cdots d\lambda_M \sum_{\lambda_{M+1}\cdots\lambda_N} |\lambda_1, \lambda_2, \dots, \lambda_N\rangle\langle\lambda_1, \lambda_2, \dots, \lambda_N|\phi\rangle. \tag{48}$$

The spectral theorem for nuclear spaces (48) thus provides the mathematical foundation for Dirac's formalism, as expressed by (20), (21), (22) and (25) as long as the topology (or convergence) in Φ can be defined such that the nuclear spectral theorem is fulfilled (e.g. as long as Φ is a nuclear space).

The standard example of an algebra of operators is the enveloping algebra $\mathcal{E}(G)$ of the Lie group G represented by τ_Φ continuous operators in the space Φ [34]. In order to represent the elements of the Lie algebra (and therewith the elements of the enveloping algebra $\mathcal{E}(G)$) by an algebra of *continuous* operators in a nuclear space Φ, one has to define the topology (or convergence) in the space Φ by the countable set of scalar products:

$$(\psi, \phi)_p = (\psi, (\Delta + 1)^p\phi), \qquad ||\phi||_p = \sqrt{(\phi, \phi)_p}; \qquad ||\phi||_0 \leq ||\phi||_1 \leq ||\phi||_2 \cdots, \tag{49}$$

where $p = 1, 2, 3, \dots$ Here Δ is defined by $\Delta = \sum X_i^2$ and is called Nelson operator and $X_i, i = 1, 2, \dots, N$, are the generators of the unitary representation of G and are also the generators of the enveloping algebra $\mathcal{E}(G)$. Then convergence of a sequence of $\phi_\nu \in \Phi$ is defined by

$$\phi_\nu \xrightarrow{\tau_\Phi} \phi \qquad \Leftrightarrow \qquad ||\phi_\nu - \phi||_p \to 0 \text{ for every } p = 0, 1, 2, 3, \dots \tag{50}$$

The Nelson operator Δ is essentially self-adjoint in Φ iff X_i are generators of a unitary representation, and the X_i are continuous operators with respect to τ_Φ-convergence. One of the possible applications of this mathematical result in physics are the spectrum generating groups G, or the Symmetry Groups S described by unitary group representations $g \to U(g)$ in $\mathcal{H}$ [34].

For a self-adjoint Hamiltonian, which is often an element of $\mathcal{E}(G)$ this leads to generalized eigenvectors of the Hamiltonian H,

$$|E\dots\rangle \in \Phi^\times \text{ with } \langle H\phi|E\dots\rangle = E\langle\phi|E\dots\rangle, \qquad \text{for all} \quad \phi \in \Phi, \quad 0 \leq E < \infty \tag{51}$$

and to time evolution for observable $|\psi\rangle\langle\psi|$ given by the unitary group $U(t) = \mathrm{e}^{iHt/\hbar}$,

$$\langle \mathrm{e}^{iHt/\hbar}\psi|E\dots\rangle = \langle\psi|\mathrm{e}^{-iHt/\hbar}|E\dots\rangle = \mathrm{e}^{-iEt/\hbar}\langle\psi|E\dots\rangle, \qquad -\infty < t < \infty, \tag{52}$$

where $\dots$ here stands for the additional quantum numbers, like the j, j_3 in (20).

4. Time evolution in quantum mechanics

4.1. Time evolution in Schwartz space Φ and in Hilbert space $\mathcal{H}$

In quantum mechanics, the time evolution of states and/or of observables is obtained by solving the dynamical equations, i.e. the Schrödinger equation for the states ϕ and the Heisenberg equation for the observables ψ. To solve differential equations, one needs boundary conditions. If one does not split H into two parts, there are two ways of proceeding[1]: One can choose the states as time independent and the observables as time dependent, this is the Heisenberg picture. Or one can choose the states as time dependent, and the observables as time independent, this is the Schrödinger picture.

In the Heisenberg picture, one chooses the Heisenberg equation as the dynamical equation for the observables $\Lambda(t)$,

$$i\hbar\frac{\partial\Lambda(t)}{\partial t} = [\Lambda(t), H]. \tag{53}$$

For the simplest case $\Lambda(t) = |\psi(t)\rangle\langle\psi(t)|$ (a projection operator on the one-dimensional space spanned by $\psi(t)$) the Heisenberg equation becomes

$$-i\hbar\frac{\partial\psi(t)}{\partial t} = H\psi(t). \tag{54}$$

The state $\rho(= |\phi\rangle\langle\phi|)$ is kept fixed. Under the boundary condition

$$\psi(t) \in \Phi = \text{Schwartz space} \tag{55}$$

the general solutions of the Heisenberg equation (54) are given by

$$\psi(t) = \mathrm{e}^{iHt/\hbar}\psi, \qquad -\infty < t < +\infty, \tag{56}$$

and the solutions of the operator equation (53) are given by

$$\Lambda(t) = \mathrm{e}^{iHt/\hbar}\Lambda(0)\mathrm{e}^{-iHt/\hbar}, \qquad -\infty < t < +\infty. \tag{57}$$

In the Heisenberg picture the state is independent of time.

In the Schrödinger picture, the dynamical equation for the state $W(t)$ is

$$i\hbar\,\frac{\partial W(t)}{\partial t} = [H, W(t)]. \tag{58}$$

For a pure state $W(t) = |\phi(t)\rangle\langle\phi(t)|$ this equation reads

$$i\hbar\frac{\partial}{\partial t}\phi(t) = H\phi(t) \tag{59}$$

for the state vector $\phi(t)$. The observable $\Lambda = |\psi\rangle\langle\psi|$ in the Schrödinger picture is kept fixed. Eq. (59) is the Schrödinger equation.

Under the boundary condition

$$\phi(t) \in \Phi = \text{Schwartz space} \tag{60}$$

[1] If one splits $H = H_0 + H$, there is also the Dirac picture.

all solutions of the Schrödinger equation (59) are given by

$$\phi(t) = e^{-iHt/\hbar}\phi, \qquad -\infty < t < +\infty, \tag{61}$$

The results (56) and (61), which are a consequence of the Schwartz space boundary conditions (55) and (60) are the standard solutions used in quantum mechanics.

If one solves the Heisenberg differential equations (54) under the Hilbert space boundary conditions:

$$\phi(t) \in \mathcal{H}, \qquad \psi(t) \in \mathcal{H} \tag{62}$$

then one obtains the following general solutions,

$$\psi(t) = \overline{U}(t)\psi(0) = e^{i\overline{H}t/\hbar}\psi(0), \qquad -\infty < t < +\infty. \tag{63-o}$$

Similarly, one obtains as solutions of the Schrödinger equation (59) under the Hilbert space boundary condition (62),

$$\phi(t) = U^{\dagger}(t)\phi(0) = e^{-iH^{\dagger}t/\hbar}\phi(0), \qquad -\infty < t < +\infty, \tag{63-s}$$

for every essentially-self-adjoint Hamiltonian $H \subset \overline{H} = H^{\dagger}$. This result (63-o), (63-s) for Hilbert space boundary conditions (62) is called the Stone–von Neumann theorem [35]. In general, the solutions for the same equation under a different boundary condition could be different. One can show [36], however, that under the Schwartz space boundary conditions (55) and (60) the solutions of dynamical equations (54) and (59) are also given by the sets of operators

$$\{U(t) = e^{iHt/\hbar} : -\infty < t < +\infty\} \tag{64}$$

and

$$\{U^{-1}(t) = e^{-iHt/\hbar} : -\infty < t < +\infty\}. \tag{65}$$

Each forms a one-parameter group, because (1) the product $U(t_1)U(t_2) = U(t_1 + t_2)$ is defined, (2) inverses exist for every element $U(t)$ and are given by $(U(t))^{-1} = U(-t)$ [2].

Thus *in Hilbert space* $\mathcal{H}$, as a consequence of the Stone–von Neumann theorem [35], *as well as in Schwartz space* Φ, for every observable $\psi(t) = U(t)\psi$ at time t there is also an observable $\psi(-t) = U^{-1}(t)\psi$ at time $-t$. This means that the probability $|(\psi(t), \phi)|^2$ to detect the observable $|\psi(t)\rangle\langle\psi(t)|$ in the state ϕ which was prepared at $t = 0$, is predicted by (64), (65) not only at a time $t > 0$ (after the state ϕ was prepared), but also at $t < 0$ (before the state ϕ was prepared). But the latter violates causality: How can a probability of an observable be measured in a state at a time $t < 0$ before the state will been prepared at $t = 0$?

Therefore, if causality is to hold one needs to solve the dynamical equations (54) and (59) under different boundary conditions such that $\psi(t)$ exists only for $t > 0$, when $t = 0$ is the time at which the state ϕ is prepared. To find (conjecture) these new boundary conditions of the Heisenberg equation (54) (or alternatively in the Schrödinger picture of the Schrödinger equation (59)), we look at the phenomenology of quantum scattering and decay.

[2] $\overline{H}, \overline{U}, U^{\dagger}$ denote the closure of the operators H, U, U^{-1} in $\Phi \subset \mathcal{H}$.

5. Phenomenology of scattering and decay

5.1. Introduction

Quantum theory falls roughly into the two categories:

Category I

Description of spectra and structure of microphysical systems. This is applied to the few stable states that exist in nature and to slowly decaying states when finiteness of their lifetime is ignored. The energy values of such states are discrete and their time evolution is assumed to be the reversible unitary group evolution. These category I quantum systems are well described by the theory using Schwartz RHS's. One does not distinguish between the set of *states* $\{\phi\}$ and the set of *observables* $\{\psi\}$ but represents them mathematically by the same Schwartz subspace Φ of the Hilbert space $\mathcal{H}$,

$$\{\phi\} = \{\psi\} = \Phi \subset \mathcal{H} \subset \Phi^{\times}. \tag{66}$$

The algebra of observables is given by an algebra of continuous operators in Φ and their time evolution (57) is described by the operator

$$U(t) = \mathrm{e}^{iHt/\hbar} = \overline{U}(t)|_{\Phi}, \qquad -\infty < t < +\infty, \tag{67}$$

where $\overline{U}(t)$ is the unitary group operator in $\mathcal{H}$.

Category II

Description of scattering, resonance phenomena and decaying states. This applies to doubly excited states of atoms (e.g. Auger states of helium), to decaying nuclei, scattering resonances, hadron resonances. For these states the time evolution is nontrivial. In the simplest case the resonances/decaying states are defined as a first-order pole on the second sheet of the S-matrix [42, 44, 45]. As a decaying state their lifetime is defined by the exponential time evolution with lifetime τ. As a scattering resonance in the cross-section they have a characteristic lineshape, (Lorentzian or Breit–Wigner) characterized by the width Γ. Neither the Schwartz RHS Φ of the Dirac formalism (66), nor the conventional Hilbert space $\mathcal{H}$ can accommodate Breit–Wigner resonances or exponentially decaying state vectors. Though the majority of physicists believe that $\tau = \hbar/\Gamma$ (or at least $\tau \approx \hbar/\Gamma$) one also knows that the mathematics of $\mathcal{H}$ (and of Φ) does not permit to define a state with exponential time evolution [37] and therefore many physicists consider this lifetime–width relation as an approximate equality.

In the phenomenological theory of scattering, one has distinguished for a long time [38] between in-''states" and out-''states". In the foundations of quantum mechanics [39, 40] one speaks of a prepared state, which is defined by a preparation apparatus and evolves according to (63-s), and one speaks of a registered observable defined by a detector which evolves according to (63-o). The Lippmann–Schwinger equations [41] use two different kets, which are boundary values from the upper or lower complex plane, second sheet of the S-matrix. In scattering theory [42] it is mentioned that in one form of the T-matrix the "incoming wave part is controlled or prepared" and in the other version of the T-matrix the outgoing wave part is controlled or known".

What all these different approaches have in common are various dichotomies: in-''state" versus out-''state" [38], prepared state versus registered observable [39], "prepared incoming wave" versus "known outgoing wave part" [42]. The precise connection between these different approaches has not been clear, since in-states and out-states have the same time evolution given by $e^{-iHt/\hbar}$, whereas states and observables evolve by different evolution groups, $e^{-iHt/\hbar}$ and $e^{+iHt/\hbar}$, respectively. Using the Hilbert space axiom (62) this division into two kinds is hard to make in a mathematical formulation in which t extends over $-\infty < t < \infty$. Neither can one do this separation using one Schwartz RHS (66).

But if one distinguishes mathematically between the two Lippmann–Schwinger kets by assigning them to two different dual spaces [43], say to $\Phi_-^\times$ for the in-kets $|E^+\rangle = |E + i\epsilon\rangle$, and to $\Phi_+^\times$ for the out-kets $|E^-\rangle = |E - i\epsilon\rangle$, then the space of prepared states Φ_- may be identified with the in-states of scattering theory and the space of observables Φ_+ may be identified with the out-states of scattering theory with inverted time evolution. With good luck the time evolution may then adjust itself such that causality (first preparation of state then detection of observables) will not be violated. This will turn out to be indeed the case.

5.2. States and observables

In the foundations of quantum mechanics a state is described by a density operator ρ or by a state vector ϕ and an observable is described by an operator $A = A^\dagger$, or $\Lambda = \Lambda^2$, or by observable vectors ψ if $\Lambda = |\psi\rangle\langle\psi|$. In a scattering experiment a state ρ (e.g. the in-states ϕ^+) is prepared by a preparation apparatus (e.g. accelerator). An observable A (e.g. the out-observable ψ^-, in scattering theory also called "out-state") is registered by a registration apparatus (e.g. detector). The quantities measured in the experiments are the probabilities to find the observable Λ in the state ρ. For time dependent problems as discussed in Section 5.1, one has two extreme choices to calculate, the Born probabilities $\mathrm{Tr}(\Lambda(t)\rho) = \mathrm{Tr}(\Lambda\rho(t))$. The left-hand side is in the Heisenberg picture with $\Lambda(t) = e^{iHt/\hbar}\Lambda(0)e^{-iHt/\hbar}$, given by (57) and ρ considered time independent. The right hand side is in the Schrödinger picture with $\rho(t) = e^{-iHt/\hbar}\rho(0)e^{iHt/\hbar}$ and Λ kept fixed. The Born probabilities are measured as ratio of large detector counts $N(t)/N$ ("relative frequencies"),

$$\begin{array}{cccc} \text{probability of observable } \Lambda & \text{is calculated as} & \text{is measured as} & \\ \text{in state } \rho & \text{Born probability} & \text{relative frequencies} & \\ & & \text{of detector counts} & (68) \\ \mathcal{P}_\rho(\Lambda(t)) & \equiv \mathrm{Tr}(\Lambda(t)\rho) = \mathrm{Tr}(\Lambda\rho(t)) \approx & N(t)/N. & \end{array}$$

For the time evolution given by the time evolution group (64) and (65) the Born probabilities in the two pictures are equal to each other and they are predicted for *all values* of $t : -\infty < t < +\infty$. According to the axiom (68) they represent the experimentally measured probabilities $N(t)/N$.

For the simplest case $\Lambda \equiv |\psi(t)\rangle\langle\psi(t)|$ and $\rho \equiv |\phi\rangle\langle\phi|$, the probability is calculated as

$$\mathcal{P}_\phi(|\psi(t)\rangle\langle\psi(t)|) = |\langle\psi(t)|\phi\rangle|^2 \tag{69}$$

for all $-\infty < t < +\infty$, if according to (67) $\psi(t) = e^{iHt/\hbar}\psi$ is given for all t.

5.3. In- and out-states of scattering theory

In scattering theory one uses in- and out-plane wave "states" $|E^+\rangle$ and $|E^-\rangle$ which fulfill the Lippmann–Schwinger equation [41–44]:

$$|E^\pm\rangle = |E \pm i\epsilon\rangle = |E\rangle + \frac{1}{E - H \pm i\epsilon} V|E\rangle = \Omega^\pm|E\rangle, \qquad \epsilon \to +0. \tag{70}$$

One speaks of complex energy, for the analytic S-matrix $S_j(E) \to S_j(z)$, for the Gamow states ϕ^{G} and for resonance poles with $z_R = E_R - i\Gamma/2$. Also the Lippmann–Schwinger equation (70) and the propagator of field theory with $z = E \pm i\epsilon$ (where ϵ is infinitesimal), suggest continuations into the complex energy plane.

Thus, on the one hand we have the in-kets $|E^+\rangle$ and the out-kets $|E^-\rangle$ of the Lippmann–Schwinger equation. On the other hand we have the prepared in-coming states ϕ and the detected out-going "state"-vectors ψ, but these are really observables $|\psi\rangle\langle\psi|$ since they are obtained ("measured") by the detector.

Combining these two aspects we conjecture that the kets $|E^+\rangle$ are the basis system for prepared in-states defined by the preparation apparatus which now we call ϕ^+. The kets $|E^-\rangle$ are the basis system for the detected out-vectors ψ^-, which we shall now call out-observables because they are experimentally defined by the registration apparatus (e.g. detector) and not like the state, by a preparation apparatus (e.g. accelerator) [46]. Thus we have two Dirac basis vector expansions, one for the in-state vectors ϕ^+ in terms of the $|E^+\rangle$,

$$\phi^+ = \sum_{j,j_3,\eta} \int_0^\infty dE |E, j, j_3, \eta^+\rangle\langle^+E, j, j_3, \eta|\phi^+\rangle. \tag{71}$$

And we have the Dirac basis vector expansion for the out-observable vectors ψ^- in terms of the $|E^-\rangle$,

$$\psi^- = \sum_{j,j_3,\eta} \int_0^\infty dE |E, j, j_3, \eta^-\rangle\langle^-E, j, j_3, \eta|\psi^-\rangle. \tag{72}$$

The $+i\epsilon$ in the Lippmann–Schwinger equation (70) indicates that the energy wave function

$$\phi^+(E) = \langle^+E, j, j_3, \eta|\phi^+\rangle \equiv \langle^+E|\phi^+\rangle = \overline{\langle^+\phi|E^+\rangle} \tag{73}$$

is the boundary value of an analytic function in the *lower complex energy* semi-plane $\mathbb{C}_-$ (for complex energy $z = \overline{E + i\epsilon} = E - i\epsilon$ immediately below the real axis).

Similarly, because of $-i\epsilon$ for the Lippmann–Schwinger equation of $|E, j, j_3, \eta^-\rangle$, the energy wave functions

$$\psi^-(E) = \langle^- E, j, j_3, \eta|\psi^-\rangle \equiv \langle^- E|\psi^-\rangle = \overline{\langle^-\psi|E^-\rangle} \tag{74}$$

are the boundary value of an analytic function in the *upper complex energy* semi-plane $\mathbb{C}_+$ and consequently it is safe to assume that

$$\overline{\psi^-(E)} = \langle\psi^-|E, j, j_3, \eta^-\rangle = \langle\psi^-|E^-\rangle \tag{75}$$

is analytic also in the lower complex energy semi-plane $\mathbb{C}_-$.

This suggests that for scattering experiments we would need not one RHS (66) but two RHS's: the RHS of prepared (in-) states $\{\phi^+\}$, defined by the preparation apparatus, we shall call Φ_- (the wave function $\phi^+(E)$ is analytic in $\mathbb{C}_-$),

$$\{\phi^+\} = \Phi_- \subset \mathcal{H} \subset \Phi_-^\times \ni |E^+\rangle. \tag{76}$$

And the RHS of detected (out-) observables defined by the detector shall be called Φ_+ (because the wave function $\psi^-(E)$ are analytic in $\mathbb{C}_+$),

$$\{\psi^-\} = \Phi_+ \subset \mathcal{H} \subset \Phi_+^\times \ni |E^-\rangle. \tag{77}$$

Physically, the energy wave functions $\phi^+(E) \equiv \langle^+ E|\phi^+\rangle = \langle E|\phi^{\text{in}}\rangle$ describe the prepared state, specifically $|\langle^+ E|\phi^+\rangle|^2$ is the energy distribution of the incident beam of a scattering experiment, it represents the property of the preparation apparatus (e.g. accelerator). The energy wave functions $\psi^-(E) \equiv \langle^- E|\psi^-\rangle = \langle E|\psi^{\text{out}}\rangle$ describe the observable measured by the registration apparatus, e.g. the detector (it represents the energy distribution of the detected particles). There is no reason that state and observable (accelerator and detector) should be described by the same mathematical representation space, as it is the case if one uses the Schwartz space axiom (66) or the Hilbert space axiom (62).

The two RHS's (76) and (77) are mathematically completely defined by defining the mathematical property of the spaces of their wave functions (73) and (74), this is the same as the correspondence (43) between the abstract Schwartz space Φ and the space of Schwartz functions $\mathcal{S}$: Φ is mathematically "realized" by the function space $\mathcal{S}$ of (43). The RHS for the prepared in-state ϕ^+ can be defined by the properties of the functions $\{\phi^+(E) = \langle^+ E, j, j_3|\phi^+\rangle\}$ and the RHS for the registered out-observable ψ^- can be defined by the properties of the functions $\{\psi^-(E) = \langle^- E, j, j_3|\psi^-\rangle\}$.

Since both $\psi^-(E)$ and $\phi^+(E)$ should be well-behaved it is natural to assume that they are Schwartz functions that can be analytically continued into $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively. Here $\mathbb{C}_\pm$ refers to the second sheet of the S-matrix where the resonance poles (and other interesting features of the S-matrix elements) are located.

5.4. Conjecturing the properties of the energy wave functions

The energy wave functions appear in the matrix elements of the S-operator. The S-matrix element of a scattering process is given by [46]

$$\begin{aligned}(\psi^{\text{out}}, S\phi^{\text{in}}) &= (\psi^{-}, \phi^{+}) \\ &= \int_{E_0}^{\infty} dE \sum_{j,j_3} \sum_{\eta,\eta'} \langle \psi^{-} | E, j, j_3, \eta'^{-} \rangle S_j^{\eta'\eta}(E) \langle^{+} E, j, j_3, \eta | \phi^{+} \rangle \end{aligned} \tag{78}$$

or in simplified notation

$$(\psi^{-}, \phi^{+}) = \int_{E_0}^{\infty} dE \langle \psi^{-} | E^{-} \rangle S_j(E) \langle^{+} E | \phi^{+} \rangle. \tag{79}$$

From the analyticity property of the $\psi^{-}(E) \equiv \langle^{-} E | \psi^{-} \rangle$ and the $\phi^{+}(E) \equiv \langle^{+} E | \phi^{+} \rangle$ follows that $\langle \psi^{-} | E^{-} \rangle \langle E^{+} | \phi^{+} \rangle S_j(E)$ can be analytically continued into the lower complex plane (second sheet), except for singularities (e.g. resonance pole positions) of the j-th partial S-matrix element $S_j(E)$. One can now try to determine further mathematical properties of the energy wave function $\psi^{-}(E)$ and $\phi^{+}(E)$ and then conjecture their precise mathematical definition and therewith the mathematical definition of the two triplets (76) and (77).

Besides the property that they are Schwartz functions that can be analytically continued into the complex plane we shall conjecture the mathematical properties of $\{\psi^{-}(E)\}$ and $\{\phi^{+}(E)\}$ from the following physically motivated requirement: (1) the pole of the S-matrix is related to the Breit–Wigner energy distribution of a resonance, and (2) the pole provides a generalized vector (ket) which has the exponential time evolution of a Gamow vector. This then associates the resonance pole of the S-matrix (1) to a Breit–Wigner (Lorentzian) line-shape of width Γ, and, (2) to an exponential decay probability of lifetime τ. Then one can compare the lifetime τ to the inverse width $\hbar/\Gamma$.

5.5. Resonances and decaying states

To determine the property of the spaces Φ_{-} (prepared states) and Φ_{+} (detected observables) we start with the heuristic properties of resonances and decaying states. The signature of a resonance in a scattering experiment is the energy dependence of the scattering cross section[1],

$$\sigma_j(E) \sim |a_j^{\eta}(E)|^2 \sim \left| \frac{1}{2i} S_j^{\eta}(E) \right|^2 \tag{80}$$

with the partial wave amplitude $a_j^{\eta}(E)$ in (80) given by

$$a_j^{\eta}(E) = a_j^{BW}(E) + B(E) = \frac{r^{\eta}}{E - (E_R - i\frac{\Gamma}{2})} + B(E). \tag{81}$$

[1]The factors between $\sigma_j(E)$, $a_j^{\eta}(E)$ and $S_j^{\eta}(E)$ depend upon conventions, and for the elastic channel there is an additional term on the right hand side of (80).

Thus $a_j^\eta(E)$ is the sum of the Breit–Wigner amplitude (also called Lorentzian)[2]

$$a_j^{BW}(E) = \frac{r_\eta}{E - (E_R - i\frac{\Gamma}{2})} \tag{82}$$

and a slowly varying background $B(E)$. A resonance $a_j^{BW}(E)$ is characterized by the two parameters (E_R, Γ) where E_R is the resonance energy (or resonance mass M_R in the relativistic case) and where Γ is the resonance width. The two parameters E_R and Γ can then be determined experimentally from the fit of (80) with (81) to the experimental line-shape data for the cross sections.

The signature of the decaying state is the exponential time evolution. Decaying states $\phi_D(t)$ are observed in process $D \to \eta$ where η are various decay products ("decay channels") described by out-vectors ψ_η. The decaying state D is characterized by its energy E_D and by its lifetime τ or the inverse thereof, the total initial decay rate $R = \frac{1}{\tau}$ (this holds if and only if the decay rate is exponential). The lifetime τ is measured by fitting the counting rate into channel η, $\frac{\Delta N_\eta(t)}{\Delta t}$ (which is proportional to the Born probability $|\langle\psi_\eta^-|\phi^G(t)\rangle|^2$ according to (68), (69)) to the exponential decay law

$$\frac{\Delta N_\eta(t)}{\Delta t} = \mathrm{e}^{-t/\tau} \tag{83}$$

(here $\Delta N_\eta(t_i)$ is the number of the decay products η registered by the η-detector during the time interval Δt).

The simplest way to derive an exactly exponential decay probability is to postulate a Gamow state vector ϕ^{G} which has the property [47]

$$H\phi^{\mathrm{G}} = \left(E_R - i\frac{\hbar}{2\tau}\right)\phi^{\mathrm{G}} \qquad \text{and} \qquad \phi^{\mathrm{G}}(t) = \mathrm{e}^{-iHt/\hbar}\phi^{\mathrm{G}} = \mathrm{e}^{-it(E_R/\hbar - i/(2\tau))}\phi^{\mathrm{G}} \tag{84}$$

and consequently its Born probabilities (69) obey the exponential law

$$\mathcal{P}_{\phi^G(t)} \equiv |\langle\psi|\phi^G(t)|^2 \sim \mathrm{e}^{-t/\tau}. \tag{85}$$

Though such a vector $\phi^G(t)$ cannot be in the Hilbert space $\mathcal{H}$ (because of the mathematical theorem [37]) and therefore can also not be in $\Phi_\mp$, it could be in one of the dual spaces $\Phi_\mp^\times$, i.e it can be a generalized eigenvector of H, like the Lippmann–Schwinger kets $|E^\mp\rangle \in \Phi_\mp^\times$.

The exponential law (85) has given such meritorious services over more than a century and has been accepted as a phenomenological law for lifetimes spanning from 10^{10} years to 10^{-10} seconds (or less). Still theoreticians and most mathematical physicists consider the exponential "law" (85) as only approximately valid.

If one is willing to accept Dirac kets and Lippmann–Schwinger kets, then there is no reason that one should disqualify a vector like ϕ^G with the property (84),

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[2]Here η denotes the species quantum numbers of various final states (channels) and is not important for our discussions of the mathematical properties here.

(85) as a "physical" vector, independent of whether it is in a Hilbert space or not. We shall therefore use the postulate of the existence of such a "Gamow" vector ϕ^G in the dual spaces $\Phi_+^\times$ and the suggestion of the Weisskopf–Wigner approximate method [44], that $\tau \approx \hbar/\Gamma$ as the starting point for the determination of the mathematical property of the RHS's (76), (77).

From the association of a resonance to the pole on the second sheet of the S-matrix $S_j(z)$ at z_R in Fig. 1 [45], we conjecture the mathematical property of the space of energy wave functions $\{\langle^- E|\psi^-\rangle\}$ and $\{\langle^+ E|\phi^+\rangle\}$, and thus fix the mathematical properties of the RHS's (76) and (77) [48]. Therewith we shall arrive at the new mathematical axiom for a quantum theory that unifies resonances and decay phenomena.

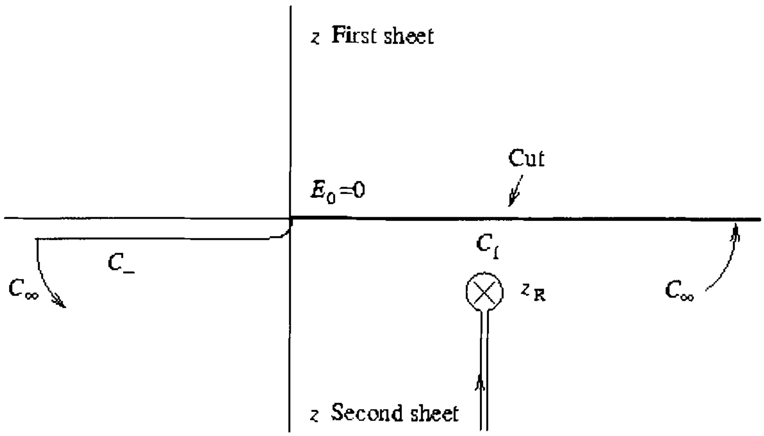

Fig. 1. Contour of integration after the contour deformation. The figure shows the second sheet around the S-matrix $S(E)$ with the cut from $E_0 = 0$ to ∞ and resonance pole z_R in the second sheet. The upper half is the first sheet $S(E)$ which one reaches when one goes across real axis between E_0 and ∞.

We start with the integral (79) for the S-matrix element and use the analyticity property of $\langle^- E|\psi^-\rangle$, $\langle^+ E|\phi^+\rangle$ and of $S_j(E)$, to deform in Fig. 1 the contour of integration from the positive real axis of the 1^{st} sheet (upper rim of the cut) into the second sheet of the lower complex semi-plane where the resonance pole is located at z_R. Then we obtain an integral along $\mathbb{C}_-$ in the second sheet, and an integral along the infinite semi-circle $\mathbb{C}_\infty$ (which must be zero and the properties of the function (73), (75) are chosen such that this is fulfilled). And we obtain the integral around each of the S-matrix poles.

For the sake of simplicity we consider here only the one pole of the S matrix around z_R. We attempt to conjecture the properties of the energy wave functions $\langle\psi^-|E^-\rangle, \langle E^+|\phi^+\rangle$ in (79), such that we derive:

1. a Breit–Wigner resonance amplitude $a_j^{BW}(E)$ (82) from the integral around the pole at z_R, and

2. that we can relate to this Breit–Wigner a Gamow ket ϕ_j^G with Breit–Wigner energy distribution

$$a_j^{BW_i} = \frac{R_i}{E - z_{R_i}}$$

$$\Longleftrightarrow \phi_j^G = |z_{R_i}, j, j_3, \eta^-\rangle\sqrt{2\pi\Gamma_i} = \int_{-\infty}^{+\infty} dE\,|E, j, j_3, \eta^-\rangle \frac{i\sqrt{\frac{\Gamma}{2\pi}}}{E - z_R}. \quad (86)$$

This can be accomplished such that Eqs. (84) and (85) are fulfilled if we make the following assumptions about the energy wave functions $\langle\psi^-|E^-\rangle$ and $\langle E^+|\phi^+\rangle$ [48]:

The function $\phi^+(E) = \langle^+E|\phi^+\rangle$ are Hardy functions [49–51] on the lower complex energy semi-plane (second sheet), and the functions $\psi^-(E) = \langle^-E|\psi^-\rangle$ are Hardy functions of the upper complex semi-plane, and therefore their complex conjugate are Hardy functions on the lower complex semi-plane.

Then $\langle\psi^-|E^-\rangle$, $\langle^+E|\phi^+\rangle$ can be analytically continued into the lower complex plane second sheet and one can in particular derive the following generalized eigenvalue equation [46, 48, 51]

$$\langle\psi^-|H^\times|E_R - i\Gamma/2^-\rangle = (E_R - i\Gamma/2)\,\langle\psi^-|E_R - i\Gamma/2^-\rangle \quad (87)$$

for every $\psi^- \in \Phi_+$. And one can also derive for values of $0 \leq t < \infty$ the following equation

$$\begin{aligned}\langle e^{iHt/\hbar}\psi_\eta^-|E_R - i\Gamma/2^-\rangle &= \langle\psi_\eta^-|e^{-iH^\times t/\hbar}|E_R - i\Gamma/2^-\rangle \\ &= e^{-iE_Rt/\hbar}e^{-t/2\tau}\langle\psi_\eta^-|E_R - i\Gamma/2^-\rangle. \quad (88)\end{aligned}$$

As a consequence of this one obtains

$$|\langle\psi_\eta^-(t)|E_R - i\Gamma/2^-\rangle|^2 = e^{-t/\tau}|\langle\psi_\eta^-(0)|E_R - i\Gamma/2^-\rangle|^2. \quad (89)$$

This can be derived only for times t with $0 \leq t < \infty$, as will be justified below in (98).

The Breit–Wigner amplitude (82) itself is obtained from the Laurent expansion of $S_j(E)$ in (79) around the first-order pole at z_R and deforming the contour of integration to $-\infty < E < +\infty$ on the second sheet (also using mathematical properties of the Hardy functions [49]). This means the complex energy eigenket $|E_R - i\Gamma/2^-\rangle$ of (86) associated to the resonance pole at $z_R = E_R - i\Gamma/2$ describes an exponentially decaying Gamow state with generalized (extending over $-\infty < E < \infty$) Breit–Wigner energy distribution (82) and with exponential time evolution (88).

5.6. Hardy space axiom of a quantum theory for resonance, scattering and decay

The idea of unifying quantum resonances and decaying states led to a new theory which distinguishes mathematically between prepared states and detected observables.

For this one uses a pair of Hardy functions[3] on the two complex energy semi-planes of the analytic S-matrix.

1. The energy wave functions of a state vector ϕ^+ are *smooth* Hardy functions analytic on the lower complex plane $\mathbb{C}_-$ (second sheet of the S-matrix),
$$\phi^+(E) = \langle^+E|\phi^+\rangle \in (\mathcal{H}^2_- \cap \mathcal{S})_{\mathbb{R}_+}. \tag{90}$$
Here $\mathcal{H}^2_-$ denotes the space of Hardy class functions of the lower complex plane and $\mathcal{S}$ is the Schwartz space, thus $(\mathcal{H}^2_- \cap \mathcal{S})_{\mathbb{R}_+})$ is the space of *smooth* Hardy functions on the positive energy axis $\mathbb{R}_+$.
2. The energy wave functions of an observable $|\psi^-\rangle\langle\psi^-|$ are *smooth* Hardy functions analytic on the upper complex plane $\mathbb{C}_+$ (second sheet of the S-matrix)
$$\psi^-(E) = \langle^-E|\psi^-\rangle \in (\mathcal{H}^2_+ \cap \mathcal{S})_{\mathbb{R}_+} \tag{91}$$

Thus, the energy wave function (73) for the prepared in-state ϕ^+ and the complex conjugate of the energy wave function (75) for the detected out-observable ψ^- of a scattering experiment are smooth Hardy functions of the lower complex energy semi-plane (2nd sheet).

These two different sets of Hardy functions $\{\phi^+(E)\} = (\mathcal{H}^2_- \cap \mathcal{S})_{\mathbb{R}_+}$ and $\{\psi^-(E)\} = (\mathcal{H}^2_+ \cap \mathcal{S})_{\mathbb{R}_+}$ are dense in the Schwartz function space $\mathcal{S}_{\mathbb{R}_+}$ and consequently in the space of Lebesgue square integrable functions $\mathcal{L}^2_{\mathbb{R}_+}$,

$$(\mathcal{H}^2_\mp \cap \mathcal{S})_{\mathbb{R}_+} \subset \mathcal{S}_{\mathbb{R}_+} \subset \mathcal{L}^2_{\mathbb{R}_+}. \tag{$92_\mp$}$$

One can show that the spaces $(\mathcal{H}^2_- \cap \mathcal{S})_{\mathbb{R}_+}$ and $(\mathcal{H}^2_+ \cap \mathcal{S})_{\mathbb{R}_+}$ are countably normed nuclear spaces [53]. Therefore the triplets of spaces

$$\{\phi^+(E)\} \equiv (\mathcal{H}^2_- \cap \mathcal{S})_{\mathbb{R}_+} \subset \mathcal{S}_{\mathbb{R}_+} \subset (\mathcal{H}^2_- \cap \mathcal{S})^\times_{\mathbb{R}_+}, \tag{93_-}$$

$$\{\psi^-(E)\} \equiv (\mathcal{H}^2_+ \cap \mathcal{S})_{\mathbb{R}_+} \subset \mathcal{S}_{\mathbb{R}_+} \subset (\mathcal{H}^2_+ \cap \mathcal{S})^\times_{\mathbb{R}_+}, \tag{94_+}$$

where $(\mathcal{H}^2_\mp \cap \mathcal{S})^\times_{\mathbb{R}_+}$ are the spaces of continuous antilinear functionals on the spaces $(\mathcal{H}^2_\mp \cap \mathcal{S})_{\mathbb{R}_+}$, form a pair of RHS's or Gelfand triplets. Consequently the heuristic Dirac basis vector expansions (71) and (72) in terms of the Lippmann–Schwinger-like kets $|E^\pm\rangle = |E, j, j_3, \eta^\pm\rangle$ are the nuclear spectral theorem [22, 23] for the two triplets of spaces:

[3]Mathematicians usually work with Hardy class functions (defined almost everywhere with respect to the Lebesgue measure) denoted by $\mathcal{H}^p_\pm$ where $+(-)$ refers to the upper (lower) complex semi-plane [49]. For the definition of $\mathcal{H}^2_\pm$ and a summary of their most important properties needed here see [51] Appendix A.2 and [52] pages 553–554.

Following Dirac and Schwartz's example, physicists work with smooth (infinitely differentiable, rapidly decreasing) functions S on the real line or $S_{\mathbb{R}_+}$ (on the positive real line). The notation $(\mathcal{H}^2_\pm \cap S)_{\mathbb{R}_+}$ then means (vaguely) that the energy wave functions admitted by the Hardy space axiom are smooth (like S) functions on the real line which can be analytically continued into the upper $(+)$ or lower $(-)$ complex semi-plane and which decrease sufficiently fast on the upper and lower infinite semicircle, second sheet of the $S_j(E)$-matrix (such the integrals over these semi-circles go to zero).

Space of prepared states:

$$\{\phi^{+}\} \equiv \Phi_{-} \subset \mathcal{H} \subset \Phi_{-}^{\times}. \tag{95}$$

Space of registered observables:

$$\{\psi^{-}\} \equiv \Phi_{+} \subset \mathcal{H} \subset \Phi_{+}^{\times}. \tag{96}$$

Here the ϕ^{+} are the vectors that describe the prepared in-states of a scattering experiment and the $|\psi^{-}\rangle\langle\psi^{-}|$ represent the observables registered by the detector of a scattering experiment. The spaces $\Phi_{\mp}^{\times}$ are the spaces of continuous antilinear functionals of $\Phi_{\mp}$. This gives a mathematical meaning to the heuristic Lippmann–Schwinger kets, they are now mathematically defined as the continuous antilinear functionals on the space $\Phi_{\mp}$, $|E^{\pm}\rangle \in \Phi_{\mp}^{\times}$ [4]. These triplets (95), (96) also give a mathematical meaning to the exponentially decaying Gamow kets: the $|E_R - i\Gamma/2^{-}\rangle$ of (89) are elements of $\Phi_{+}^{\times}$.

One now needs to solve the dynamical equations, the Heisenberg equation (54) for $\psi^{-}(t)$ and the Schrödinger equation (59) for $\phi^{+}(t)$ under the new boundary condition

$$\psi^{-}(t) \in \Phi_{+} \qquad \text{and} \qquad \phi^{+}(t) \in \Phi_{-}. \tag{97}$$

For the Hilbert space boundary conditions (62) one obtained from the Stone–von Neumann theorem [35] the unitary group evolutions (63-o) and (63-s). Similarly, for the Hardy space boundary conditions of the same dynamical equations (54) and (59) the time evolutions are given according to the Paley–Wiener theorem [54] by the following two semi-groups:

For the Heisenberg equation of the observables in the space Φ_{+}:

$$\psi^{-}(t) = \mathrm{e}^{iHt/\hbar}\psi^{-}, \qquad t_0 = 0 \leq t < \infty. \tag{98}$$

For the Schrödinder equation of the states in the space Φ_{-}:

$$\phi^{+}(t) = \mathrm{e}^{-iHt/\hbar}\phi^{+}, \qquad t_0 = 0 \leq t < \infty. \tag{99}$$

A special case of this semi-group time evolution is the time evolution of Gamow ket (88), (89). The semi-group limitation $0 \leq t < \infty$, which followed from the Paley–Wiener theorem, is now a welcome feature since it avoids the "exponential catastrophe" for the Born probabilities $|\langle\psi_{\eta}^{-}(t)|E_R - i\Gamma/2^{-}, j, j_3, \eta^{-}\rangle|^2$, which would ensue if time evolution were derived as a unitary group evolution (64), (65) from the Hilbert-space or Schwartz-space boundary condition (62), (60), respectively.

In contrast to Schwartz space axiom (60) and the Hilbert space axiom (62), the new Hardy space axiom (95), (96) distinguishes also *mathematically* between the prepared states $\{\phi^{+}\}$ and the detected observables $|\psi^{-}\rangle\langle\psi^{-}|$, which are physically different prepared states (preparation apparatus, accelerator) for the ϕ^{+}, and registered

[4]The mismatch of the $^{\pm}$ and $_{\mp}$ is due to the difference in notation of the physicists for the Lippmann–Schwinger kets $|E^{\pm}\rangle$ and in the notation of the mathematician for the Hardy spaces $\Phi_{\mp}$ (or $\mathcal{H}_{\mp}$).

observable (registration apparatus, detector) for the ψ^-. The axiom (95), (96) replaces the Hilbert space axiom (62) of standard quantum mechanics and it also replaces the Schwartz space axiom (66), which provided the mathematical theory of Dirac's bra-and-ket formalism.

The new axiom (95), (96) provides the mathematical foundation of a quantum theory that includes many of the heuristic notions of quantum scattering and decay. It overcomes the inconsistencies connected with deviations from exponential decay and it unifies resonance scattering and decay phenomena with $\Gamma = \hbar/\tau$ as an *exact relation*. But (98) predicts time asymmetry for the Born probabilities

$$P_\phi(\psi(t)) = |\langle\psi^-(t)|\phi^+\rangle|^2, \qquad \text{for } t \geq 0 \text{ only,} \tag{100}$$

and many people may be uncomfortable with this "microphysical irreversibility". The prediction of a semi-group time evolution (98), (99) also raises two questions: What is the physical meaning of a distinguished semi-group time $t_0 = 0$; and how does one observe the time at which the state ϕ has been prepared and the registration of an observable $|\psi(t)\rangle\langle\psi(t)|$, as Born probability $P_\phi(\psi(t)) = |\langle\psi(t)|\phi\rangle|^2$, can begin?

6. Epilogue

We have surveyed the development of the mathematical tools of quantum physics, from numbers to matrices and to operators on infinite-dimensional spaces. Starting with a linear space, we progressed to the Hilbert space, and then to the Schwartz space triplet for Dirac's formalism, finally we arrived at a pair of Hardy space triplets describing a time asymmetric quantum theory of scattering, resonance and decay phenomena. Some of these steps are well accepted, others are still debated and one may ask the following questions: 1. Does one have to go from one Hilbert space to a triplets of spaces and then even to a *pair* of such triplets of spaces? 2. Are these Hardy space triplets just mathematical beautifications or do they actually provide new physical insight?

The first question has been answered by the development in the teaching of quantum mechanics. As documented by the recent textbooks of quantum mechanics, physicists do not use the Lebesgue integration as *needed* for the Hilbert space, instead they simply use the Dirac formalism. This requires one Schwartz-RHS to give Dirac's formalism a mathematical meaning.

But then, can one not just use Dirac's formalism? For simple algebraic calculation of branching ratios and discrete energy values one could. The Schwartz-RHS of the Dirac formalism [26–34] would be sufficient to describe structure and discrete spectra of quantum systems.

But this Schwartz-RHS of the Dirac formalism is not sufficient for a full description of quantum scattering, resonance and decay phenomena, because it cannot provide a mathematical theory for the Lippmann–Schwinger kets and Gamow vectors. One needs a generalization of Schwartz's distribution theory to the RHS's which allow analytic continuations to complex energies and one needs a pair of RHS's in order to distinguish between prepared states (in-states) and detected observables (modified "out-states").

Most physicists believe that resonances and decaying states are somewhat the same and that the Breit–Wigner (or Lorentzian) width Γ and the exponential lifetime τ are at least approximately the inverse of each other: $\tau \approx \hbar/\Gamma$. Only recently, has the relation $\tau = \hbar/\Gamma$ been confirmed to a high degree of accuracy in two separate (nonrelativistic) experiments [55], [56] in which both the lifetime τ and width Γ of the same decaying state have been measured independently to such a high degree of accuracy that $\tau = \hbar/\Gamma$ can be considered as established beyond the accuracy of the Weisskopf–Wigner approximation.

Neither in the Schwartz space triplet, nor in the Hilbert space can one define an exponentially decaying state. For Gamow kets, for S-matrix poles and for Lorentzian line-shape one needs analytic continuation; the mathematics that was ready-made for this is the pair of Hardy spaces. One just needs to modify the Schwartz-RHS slightly to arrive at a *pair* of Hardy-RHS's, one for prepared in-states and the other for the detected out-observables (usually misunderstood as out-states) of scattering experiments. Statements like $\tau \approx \hbar/\Gamma$, which is not obtained in Hilbert space quantum mechanics, can be *derived as exact relations* $\tau = \hbar/\Gamma$ for kets of the Hardy-RHS.

Most of the new mathematical entities introduced by the Hardy-RHS had already a heuristic counterpart. One just needed to put them together correctly and give them a mathematical meaning. Then one could even extend this theory to the relativistic domain [46, 57] and obtain new physical results.

For instance, one could answer the question: What is the right value of the mass and the width of the (relativistic) Z-boson-resonance and specify therewith one of the most fundamental and most accurately measured quantities of particle physics. Since there was no theory of relativistic resonances and thus not an accurate line-shape formula, like the nonrelativistic Breit–Wigner, the propagator definition in the on-the-mass-shell-renormalization scheme (though not gauge invariant) was used for the definition of the mass and width (M_Z, Γ_Z) of the Z-boson, and the mass value was determined as $M_Z = (91.1875 \pm 0.0021)$ GeV [58]. If one uses instead the relativistic Breit–Wigner line-shape formula with (M_R, Γ_R), based on the prediction $\tau = \hbar/\Gamma_R$ from the Hardy space axiom [59], one obtains for the mass of the Z-boson $M_R = (91.1161 \pm 0.0023) GeV$, which is significantly different from M_Z. The same argument applies to the other well measured relativistic resonances like Δ and ρ, for which several different definitions have been in circulation, leading to a multitude of different mass definitions ("Breit–Wigner masses" versus "pole position masses" [58]) with significantly different values for the fitted mass values for these different definitions [60]. The Hardy space axiom chooses the S-matrix pole definition.

Thus, such fundamental mathematical choice whether to use the Hilbert space or the Hardy-space axiom, has a direct bearing upon the numerical values which have required so much effort and expense to be experimentally determined to a high accuracy.

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