A Course in Mechanics

Lectures given for IMSc Batch of 2012

University of Hyderabad

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Contents

Preface

These lecture notes follow the lectures delivered to the first semester students of the Integrated Masters Program in Science of University of Hyderabad.

The class consisted of about sixty four students who would later in the third year branch out to their Masters' program in different subjects of science, viz, Physics, Chemistry, Mathematics and Systems Biology. Several students had very little or no exposure to differential and integral calculus and differential equations. Many of them, though had used vectors, but were not comfortable with vector algebra and analytical geometry.

All the students had an exposure to the Newtonian mechanics in their 12th class. But had varying levels of confidence in applications to standard problems. Some of the students were very enthusiastic and prepared to take on new and challenging topics.

The goal that was set for the design this course was modest one to raise bring every one to a common, higher level and at the same time to avoid repetition and provide new insights and to provide complete picture of new topics such as non inertial frames and rigid body dynamics.

The course was planned (three lectures per week) to consist of five major components.

1. Mathematical preparation
2. Review of Newtonian Mechanics
3. Non-inertial frames
4. Conservation laws and Applications
5. Rigid body dynamics

The applications of Newtonian mechanics was taken up mostly during the tutorial sessions only. In addition to this, there problems related to the topics covered in the lectures were also included as part of tutorials.

All tutorial sets, other assignments, test and examination papers are being mad available at the end. The solutions, not available at this time, may be made available in near future.

All lectures were recorded on a digital voice recorder and snap shots of blackboards were taken. These were then used to produce the lecture notes very close to the lectures actually delivered to the class. Only a slight For this reason this set is not a polished and finished product. For example, while delivering the lectures sometimes it was found necessary to repeat an earlier topic, it appears in the notes as well in the same fashion. Ambiguities/errors in assignments and other course material have been brought to my notice. A revised and corrected version of the course will be made available in near future.

As the audio recordings of most of the lecture sessions was available it was decided to include questions by students, their responses to my questions and the discussions during the lectures. This is a unique feature of these lecture notes.

I thank all my students for their support and participation in the course. By sheer presence in the class, if nothing else, each and every one has contributed to the planning of the course and preparation of these lecture notes. I thank them for making teaching enjoyable and specially for turning the last few minutes of this course on the very last day in the University of Hyderabad a memorable experience.

Special thanks are due to Vijay Kumar who provided valuable assistance during the tutorial sessions,for his active involvement in running of the course and suggestions regarding all aspects of lectures and course content. I thank K.P.N. Murthy for making a visit to Birla Science Museum a reality. I also thank the staff of Centre for Integrated Studies for all the support they provided during the course.

Lesson 1

Introduction, Course Plan and All That

July 23, 2012

§1 Welcome

WELCOME TO UNIVERSITY OF HYDERABAD I extend a warm welcome to all the students of the Integrated Masters in Science of University of Hyderabad.

This course on Mechanics will cover basics of Mechanics. For details see the outline of the plan given below.

The lecture notes for the course will be made available on a course site and also on Web Site for Physics Resources.

Please send an email to `vijaykumarsop@gmail.com` with one line of the following information in the body of the email.

RollNo, RollNo, firstname, lastname,email address

if you have not already done. This will be needed to create an account for you to log in the Course Site.

If an account on the course site has been created for you, please login and change your password immediately. If your account does not does not exist, or you have any problems, please contact me.

§2 Aims and Objectives

- ⊙ To consolidate what you have already learnt in 12th class.
- ⊙ To provide a short introduction to mathematics topics needed for the course. The details and usage will be explained as and when a topic is needed.
- ⊙ To give a complete picture of areas where ever possible with as little demand as possible on mathematical prerequisites.
- ⊙ To upgrade technical levels by using vectors, matrices, complex numbers and other mathematical tools already available.
- ⊙ To encourage group activities.
- ⊙ To bring advanced things down to your level and make them accessible to you.
- ⊙ Every one should get equal opportunity to participate in class room discussions.
- ⊙ Make a conscious effort to ensure that no one should be left behind.
- ⊙ Emphasis will be on understanding of basics concepts and problem solving.

§3 Prerequisites

1. A working knowledge of Vector Algebra.
2. Use of Coordinate Geometry.
3. Knowledge of Differential and Integral Calculus will be needed.
4. Some knowledge of matrices will be helpful.

§4 Main Content of the Course

The first twelve chapters of H.C.Verma's book form main content of the course. Problems for tutorials and examples in the class will be mostly from the books by H.C. Verma(HCV) and Halliday Resnick (HR). The plan of the course is as follows.

1. Review of Vector Algebra
2. Review of Complex Numbers
3. Review of Coordinate systems and Analytic Geometry
4. Review of Differential and Integral Calculus
5. Newton's Laws of Motion

6. Conditions for Equilibrium
7. Applying Newton's Laws, Equation of motion in one, two and three dimensions. Projectile, Friction, Motion in Gravitational Field;
8. Conservation of Linear Momentum
9. Work and Energy, Conservation of Energy
10. Examples and uses of Conservation Law
11. Central force problems; Kepler's' Laws
12. Non-inertial frames
13. Kinetic energy, Angular Momentum of a system of particles, Moment of Inertia. Torque, Conservation of Angular Momentum
14. Rigid Body Dynamics, Gyroscope

§5 Beyond text books such as HCV and RH

The following list will give you some idea about topics to be covered beyond the two text books HCV and HR.

1. Inertial and non-inertial frames; Pseudo forces.
2. Rotation of axes and transformation of position vector of a point.
3. Motion in rotating frames. Coriolis force and centrifugal force. Effects of Coriolis force.
4. Kinetic energy and angular momentum of a rigid body; Moment of inertia tensor
5. Rigid Body Dynamics including general motion of rigid bodies, Euler's Equations
6. Kepler's Laws; Proof of orbits being planer and elliptic orbits.
7. Small oscillations about equilibrium in one and two dimensions.

These topics do not require any new mathematical ability and only a small amount of preparation will be needed. The above mentioned topics are usually not a part of a standard B.Sc. course. How much of all this can actually be covered will depend on how the class responds.

§6 Miscellaneous

The success of the course depends on your participation during the lectures and tutorial sessions. Please feel free to interrupt and ask questions in the class. The questions on prerequisites are also welcome.

§7 Contact information:

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Telephone: University Exchange: 4355; 040-23134355;

Mobile: 96189 32066

email: akksp@uohyd.ernet.in; akkhcu@gmail.com

You will yourself discover some more ways of contacting me.

Lesson 2

Evaluation Process

July 23, 2012

§1 Evaluation

The evaluation will consist of Final examination and continuous assessment units as follows. The continuous assessment carries a weight of 40% and the final examination will carry a weight of 60%.

§2 Final Examination

will consist of a written examination of three hours. This will carry a weight of 60%. The question paper will have a fair mix of derivations from class, problems and short answer type questions.
You are allowed to bring in one your own A4 sheet of formulas and important results.
You are not permitted to borrow, lend or exchange the formula sheet from your friends.

§3 Tutorial

- ⊙ It is expected that this set is solved in the class and submitted on immediately after the class.
- ⊙ You are encouraged to discuss with me and the TA in the class.
- ⊙ These problem sets will be checked, graded and returned.

§4 Continuous Assessment

The continuous assessment will carry a weight of 40% and will consists of several units of 20 marks each such as Class Tests, Tutorials, Quiz, Group Activities.

§4.1 Tutorial

The following points about the tutorials be noted.

- One lecture per week will be reserved for tutorials.
- ⊙ The tutorials will be held in the Monday afternoon class.
- ⊙ A set of problems will be given either in advance or on the day of tutorial.
- ⊙ It is expected that this set is solved in the class and submitted immediately after the class.
- ⊙ You are expected to discuss with problems with your friends or with any one whom you wish and come prepared.
- ⊙ In the tutorial hour, you are encouraged to ask me, and other tutors who may be present, about your difficulties .
- ⊙ These tutorial sets will be checked, evaluated and returned.
- ⊙ Make best use of the class time during tutorial sessions and grades will take care of themselves.

§4.2 Class Tests

- ⊙ There will be two written class tests for one hour each. Each test will form one unit of assessment. The test will consist of problem solving and questions requiring short answers. Derivations will not be asked in the tests
  You are allowed to bring in one your own A4 sheet of formulas and important results. You are not permitted to borrow, lend or exchange the formula sheet with any one during the tests.

§4.3 Quiz

- ⊙ Quiz will be mostly unannounced. Each quiz will be of 15 - 30 minute duration and the answers will be discussed immediately after the quiz is over.
- ⊙ You are allowed to consult your own class notes during the quiz.
- ⊙ No extra time will be allowed during the quiz at any cost.
- ⊙ There will be quizzes of different variety. These will include problem solving, objective type, grading the response type etc.

§4.4 Group Activity

- ⊙ Groups will be formed and announced for group activities.
- ⊙ The aim of group activities is to encourage discussions among the students.
- ⊙ Details of mode of group activity will vary from one to another group activity and will be announced separately.
- ⊙ One mode of group activity will be giving feedback to me. For every topic covered you will be asked to summarize
  - (a) Prerequisites
  - (b) Key concepts
  - (c) Important facts and results
  - (d) Skill acquired by you.
  - (e) Your confidence level on a scale of 0 to 5.
- ⊙ Each member of group must individually write summary on one A4 sheet. The group must staple the answers and submit with Name, Group Id and Roll Number and Date of Submission prominently mentioned .
- ⊙ The "Summary Activity" is compulsory for every one. It will be graded on a scale -5 to +5. It will carry a negative marking for no submission and zero will be awarded for not putting in enough effort.

§5 Evaluation

- There will be two tests and one final examination. In addition quiz, or other assignments may be given from time to time.
- The Final examination will be of three hour duration carries a weight of 60
- The three minor tests will be of one hour duration and will be graded out of 20 marks. The best two of three tests will be given credit. This will make 20% of the total marks for the course.
- All tutorials will be graded out of 20 marks and will carry 10% weight.
- Other assignments will carry a weight of 10%. This will include 5% for overall response/performance in the class. More about it will be discussed in the class.
- Emphasis will be on both understanding of basics concepts and problem solving.

§6 Grading

- There will be five units of assessment. Two units will be class tests, one unit will consist of all tutorials. The other two units will be quizzes and group activities.
- Best 6 six out of 5 or more evaluations,(or best 3 out of a total 4, or best 2 of 3) evaluations in each unit of assessment will be included for final grading. Leaving out marks in one or two where the performance is lowest, the total marks secured in each unit will be added up and scaled to maximum of 20. .
- The total of marks secured in the best 3 of the five units of assessment will be scaled to 40 and will form the marks for internal assessment.
- The marks for continuous assessment and final examinations will be added and grades will be awarded according to an approved table for School of Physics Courses, Click here for the table
- The evaluation process will be kept fully transparent. All papers of internal assessment will be returned and you will be welcome to see your final examination papers at the end after evaluation process is over.
- All the details of internal marks as well as final marks will be made available on Internet.

Lesson 3

Review of Vector Algebra

July 25, 2012

§1 Vectors as Geometric Objects

Recall that the vectors are introduced as 'quantities' that have both length and direction. The vectors can be represented by a line with an arrow. The length of the line represents the magnitude and the direction is represented by the direction in which the arrow points. This representation of vectors is very useful in two dimensions.

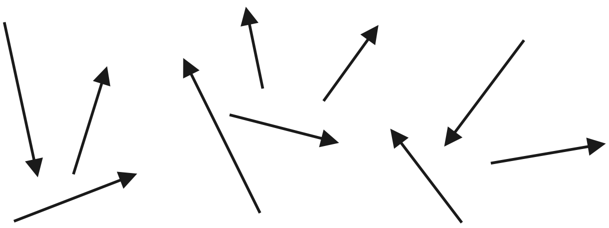

Fig. **1.** Directed lines as vectors

Remarks:

- ⊡ Two vectors, drawn from different base points, also called the root, are equal if they are parallel and their lengths are equal.
- ⊡ The above definition of equal vectors does not distinguish between vectors drawn from different roots. For most applications in mechanics this is sufficient; but not for rotational motion. For example, the point of application of a force (*i.e.* the root) is important, and is needed for computation of torque of a force.

In three dimensions the usefulness of the above representation is limited by difficulties related to visualization.

We need to represent vectors normal to the plane of paper. A normal to the plane of paper can have two directions, a) pointing outwards and b) pointing into the paper. The symbol $\odot$ will represent direction of normal out of the paper and the symbol $\otimes$ will represent the direction perpendicular and into the plane of paper. To remember this think of *times* as "into" and the "." (dot in $\odot$) as an "o" (for out) of a very small radius

Thus for example, if the x and y axes of a coordinate system we use are in the plane of paper, as shown below, the z axis will be outwards.

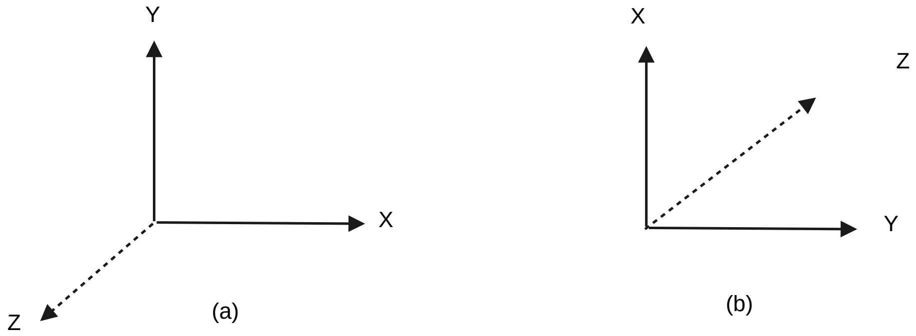

Fig. **2.** Right handed coordinate system (a) Z-axis points $\odot$, out of the plane (b) Z-axis points $\otimes$, into the plane

Some definitions and properties

- We denote the vectors as $\vec{A}, \vec{B}, ...$ etc., their lengths as $|\vec{A}|, |\vec{B}|, ...,$ or simply by $A, B, ...$

- A **unit vector** is a vector of unit length and will be represented as $\hat{n}$. Thus $|\hat{n}| = 1$.

- When a vector $\vec{A}$ is multiplied by a constant α, the magnitude of the resulting vector equals the original value multiplied by the absolute value of the constant α.

$$|\alpha\vec{A}| = |\alpha||\vec{A}| \tag{3.1}$$

- Multiplication of a vector $\vec{A}$ by a positive constant gives a vector in the same direction as $\vec{A}$, whereas multiplication of the vector $\vec{A}$ by a negative constant gives a vector pointing in direction opposite to that of $\vec{A}$.

- A vector of zero length, written as $\vec{0}$ or simply as 0, will be called **null vector** or **zero vector**. It must be noted that null vector cannot be assigned any direction. So we shall use 0 to denote the null vector as well as the real number zero. It will always be clear from the context whether 0 in a given situation stands for the number or the null vector.

- Multiplication of an arbitrary vector by 0 produces a null vector, $0\vec{A} = 0$.

- If $\vec{A}$ is a non-zero vector, multiplication of $\vec{A}$ by $1/|\vec{A}|$ produces a unit vector in the same direction which will be denoted by $\hat{A}$. Thus

$$\hat{A} = \frac{1}{|\vec{A}|}\vec{A} \Rightarrow |\hat{A}| = 1; \text{ also } \vec{A} = |\vec{A}|\hat{A}. \tag{3.2}$$

  Thus if $\vec{n}$ is a unit vector in the same direction as a given vector $\vec{A}$, then $\vec{A} = |\vec{A}|\hat{n}$.

Examples of multiplication of vectors by a constant are shown in figure below.

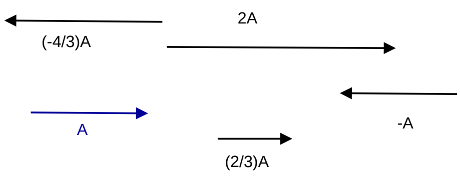

Fig. **3.** Multiplication of vector A by different constants

Sum of Vectors

The **sum of two vectors** is given by the **parallelogram rule**: *If the given vectors are drawn from a point, their sum is a vector given by the diagonal of the parallelogram with the two vectors as its sides*, see Fig.4 below. Result of addition of two or more vectors can be obtained in several ways as shown in figures below. In general, to add vectors we draw a vector then root of the next from head of the first vector, the third one is drawn with its root from the head of second one and so on. The sum of all vectors is the given by the vector by dirceted line from the root of the first vector to the head of the last vector.

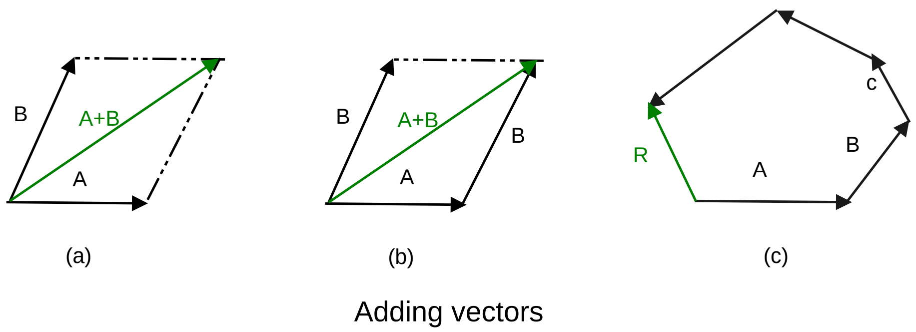

Fig. **4.** (a) Paralleogram rule (b) Triangle rule (c) Using polygon for several vectors

It is easy to see from this definition that the order of adding vectors is unimportant. The result of addition $\vec{A} + \vec{B}$ is same as that for $\vec{B} + \vec{A}$.

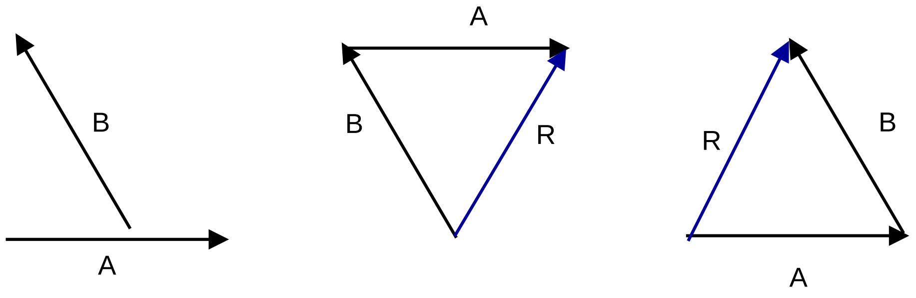

Fig. **5.** Order of addition is not important for vectors (a) Given vectors (b) R=B+A (c) R=A+B

This property of operation of adding the vectors has a name; we say that the law of vector addition is **commutative**.

§2 Properties of operations on vectors

The operation of adding two vectors and the operation of multiplying a vector by a number (henceforth we will call it a scalar) has been defined. Thus we know that two vectors can be added and that a vector can be multiplied by a scalar. It is not very difficult to see that these operations satisfy the following properties.

1. Given two vectors $\vec{A}, \vec{B}$ their sum $\vec{A} + \vec{B}$ is also a vector
2. The vector addition satisfies the property

$$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}. \tag{3.3}$$

This property is known as **associative property** and is illustrated in Fig.6. below.

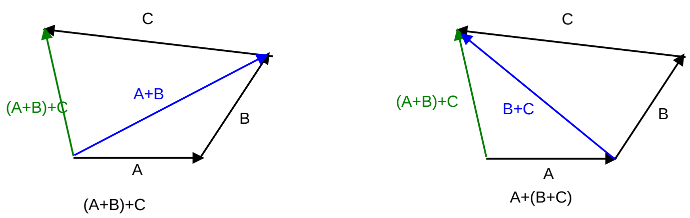

Fig. **6.** Associative Nature of Vector Addition

The importance of this property is that when we want to add several vectors and talk about a sum $\vec{A}_1 + \vec{A}_2 + \vec{A}_3 + \cdots + \vec{A}_n$, we need not put brackets to specify the way they are to be combined. In particular, the brackets in an expression such as that appearing in Eq.(3.3) can be dropped.

3. The null vector $\vec{0}$ has the property that

$$\vec{A} + \vec{0} = \vec{A} = \vec{0} + \vec{A}. \tag{3.4}$$

   and we say that *the null vector is the* **identity element** *for vector addition.*

4. For a given vector $\vec{A}$, there is always a vector $-\vec{A}$ such that the two add to give the null vector.

$$\vec{A} + (-\vec{A}) = \vec{0} = (-\vec{A}) + \vec{A}. \tag{3.5}$$

   One says that *every vector $\vec{A}$ has an (additive)* **inverse** given by $-\vec{A}$.

In addition to the above properties, the following properties can also be seen to hold.

1. $(\alpha + \beta)\vec{A} = \alpha\vec{A} + \beta\vec{A}$
2. $\alpha(\vec{A} + \vec{B}) = \alpha\vec{A} + \alpha\vec{B}$
3. $(\alpha\beta)\vec{A} = \alpha(\beta\vec{A})$
4. $0\vec{A} = \vec{0}$

The above properties hold for arbitrary vectors $\vec{A}, \vec{B}, ..$ and arbitrary real numbers $\alpha, \beta, ...$

⊡ The properties listed are more or less obvious one. Then why do we list them here? One of the important activities of the mathematicians is to make use of selected results in an area to define new abstract concepts. Why is activity useful ? It is useful for several reasons. One of the reasons is that one can take over the results in one area to another area which may appear to be totally unrelated. Thus the properties listed above are the *starting point* for introducing *vector spaces* in an abstract setting.

LastUpdate: November 12 ,2013

Lesson 4

Newton's Laws

July 27, 2012

§1 Introduction

Work of numerous scientists, well known not so well known, has resulted in formulation of laws for many major areas of Physics. In each case the laws are small in number and have a large domain of applicability. Most applications and results in these areas can be derived from, or traced back to, a small number of basic laws or postulates. For electromagnetic theory the Maxwell's equations play the central role, for quantum mechanics a small number of postulates can be listed and so is the case with thermodynamics, statistical mechanics. In mechanics the three Newton's law play pivotal role from which everything else can be derived.

§2 Newton's Laws of Motion

First Law A body continues to remain in state of rest, or in state of uniform motion unless acted upon by an external force.

Second Law The rate of change of momentum of a body is directly proportional to the applied force.

Third Law Every action has equal and opposite reaction.

It is important to note that the results needed for application in mechanics can be reduced to, or derived from, the above laws. Before we take up applications, there are several important points to be discussed about the three laws. A quick list of points of discussion is

1. Coordinate system

2. Frames of reference

3. Use of vectors, Calculus
4. Uniform motion
5. Applications to equilibrium of rigid bodies
6. Application to dynamics of systems of point particles
7. Inertial frames, Gravitational and inertial mass, Domain of Applicability, Beyond Newtonian Mechanics, Limitations of the Newtonian formalism and Laws,...

We shall take up the above issues one by one, though not necessarily in the order given here. A discussion of some of the issues mentioned will be taken up in due course of lectures.

Frames of reference

The laws have been arrived at by long analysis, detailed of experimental observations, mostly concerning the motion of heavenly bodies. The method of scientific analysis demands that every theory that is proposed must be verifiable and that a continuous effort must be made to check the predictions against experiments, paying due attention to discrepancies if any, no matter how small it may appear to be. Efforts to understand disagreement between theory and experiments have led to revolutionary changes in theories and concepts of Physics.

The laws by themselves are not subject to a direct verification by experiments. Any experiment on a physical system would give a set of numbers to be compared against predictions of theory. In order to carry out this job meaningfully, it is necessary to have a convenient way of describing physical systems. These descriptions are often idealised models which provide a good description to a great accuracy. Thus, for example, an "approximation" of treating the Earth and the Sun as point objects describes the orbit of the Earth very accurately. When we begin to pay attention to finer and finer details and make an attempt to understand them, we must refine the model also. To explain the tides, for explain one must one cannot obviously approximate the system consisting of the Earth, and the sea as a point particle or a rigid body.

- ⊡ In most part of this course on mechanics we shall be concerned with motion of object which are to be treated as point particles. Later we will discuss the motion of rigid bodies.
- ⊡ To describe a the position of a point particle, *at a given time*, we need to select a **coordinate system.** The position and velocity of a particle completely describe a particle at a given time and we say that the **state** of particle is known if we know the velocity and the position.
- ⊡ Suppose we know the positions and velocities of all the particles at *one time*, is our job over? No, far from it, in order to carry out the business of recording observations, derive laws, make predictions about the motion of a particle, it is necessary that they

be recorded at different times. In addition to measurement of position coordinates, *we must add a clock* to as a time measuring apparatus. A choice of coordinate axes, a way of measuring coordinates, and a clock to measure time intervals, all this put together constitutes a **frame of reference**.

In the above we have discussed the need for introducing a frame of reference. Let us now talk about vectors. This course will make a *heavy use of vectors.* Why do we want to complicate our lives by introducing extra notation of vectors? Does it serve any useful purpose?. After all, it seems that it is just another way of writing good old equations in a new notation. Nobel Laureate, and famous Physicist, Abdus Salam has once remarked, in context of *development renormalisation in Quantum Field theroy*, the main effort required was in finding a good common notation that every one understood. Hence a good notation has a lot of advantages in communication of our ideas to others. Use of vectors in mechanics serves this purpose.

However, vectors appear in mechanics, in fact everywhere in Physics, for more important reasons. Let us recall that vectors are objects with three components. The use of vectors replaces three equations by a single vector equation. However, the use of vectors as objects with three components ties the *description and discussion* to the coordinate system selected because actual values of the components of a vector depend on the coordinate system chosen. Thus, it is expected that results obtained by different observers should be related in a definite fashion and the laws derived from experiments must be independent of the choice of coordinate system. If this was not the case, there would be a situation in which every one will have his, or her, own laws of Physics which depended on the choice of coocrdinate system and there will be chaos. *The Nature does not care about the coordinate systems, the fundamental laws of Physics do not refer to any special choice either.* How do we then ensure that the equations, that we writes, give us teh same Physics in different frames? Do we have to verify every time that our equations of Physics satisfy the requirements of yielding the results which are independent of choice of frame of reference? The answer, in one sentence, is that the use of vectors ensures that no further discussion is needed and invariance of laws, and of equations, of Physics becomes manifest when we use vectors. Let me emphasise that this is not an issue that appears in mechanics alone. This is a question relevant to all major areas of Physics. For example, it is very helpful to use 4-vector notation when dealing with relativistic theories.

So let us accept that the use of vectors is inevitable. How does the use of vectors help us when the components of vectors are dependent on the choice of coordinate system? The answer to this question is that use of components is not essential for vectors. The vectors can be thought of a geometrical objects having a magnitude and direction and all operations on vectors them can be formulated without introducing any coordinate system. This frees us from the burden of choosing a coordinate system to write equations; the laws can be translated in terms equations that do not refer to a particular coordinate systems. *The form of equations retains the same form in all the frames of reference; no further work is then needed to prove that the equations have desired property of giving results that are independent of choice of frames.* Physicists have a compact way of describing this happy situation, and they say that the equations are **manifestly form invariant**.

§3 Key concepts

Several new concepts have been introduced for the first time in this lecture. Some other concepts will be familiar to you form your college days. A list of the concepts that have appeared is given below.

- Position, Velocity, Momentum,
- State of a Physical System
- Coordinate System, Frame of reference
- Vectors, scalars, tensors

You need to learn these concepts as clearly as you can. More discussion will follow in later lectures. My policy is to introduce introduce *important* concepts as early as possible. It is neither my intention, nor is it possible, to have an exhaustive discussion about everything at the time of introduction; however, in these lectures, we make a conscious effort to go a little, *only a little*, beyond the the boundary, *the Lakshman Rekha*, that we ourselves draw by sticking to what is actually needed at a given stage. One such example, of a new concept, is the concept of **state of a physical system**. Strictly speaking, we can *live* without the concept of state which is not needed in this entire course and is not introduced in most books of mechanics at undergraduate level; but it is present everywhere and eventually, as you learn more and more, it becomes necessary to talk about it; for example, when one starts doing quantum mechanics. Please remeber that it is sufficiently important, easy to understand and to begin using. It is recommended that you make it part of your standard Physics vocabulary.

Lesson 5

Co-ordinate Systems

July 30, 2012

§1 Matrices

An array of real or complex numbers with m columns and n rows is called $m \times n$ **matrix**. For example

$$X = \begin{pmatrix} 1 & 2 & -4 \\ -3 & 2 & 7 \end{pmatrix}$$

is a 2×3 matrix. A number in i^{th} row and j^{th} column is called the ij element of the matrix; for a matrix S the ij element will be denoted by A_{ij}. Thus 2,3 element of the above matrix X is 7 and the 1,3 element is -4. The addition of two $m \times n$ matrices A, B both having m row and n columns can be defined and the result of addition is a similar matrix C, and we have $C_{jk} = A_{jk} + B_{jk}$.

The multiplication AB of two matrices A, B is defined only if the number of columns of matrix A is equal to the number of rows of the matrix B. If $X = AB$, then

$$X_{ij} = \sum_k A_{ik} B_{kj}$$

Please refer to the discussions in the class for explanation of rules for matrix multiplication. A matrix having only one column will also be called **column vector** and a matrix with one row will called a **row vector**.

Matrix Representation for Cross Product and Triple Product A useful form for the cross product of two vectors, and the triple scalar product of three vectors is the determinant form

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \tag{5.1}$$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \tag{5.2}$$

$$\tag{5.3}$$

§2 Plane Polar Coordinates

Instead of using the Cartesian coordinates, one may specify the position of a point in a plane by means of **plane polar coordinates** (r, θ) as shown in the figure. The relation between the Cartesian (x_1, x_2) and plane polar coordinates (r, θ) is given by

$$x_1 = r\cos\theta \qquad x_2 = r\sin\theta \tag{5.4}$$

$$r = \sqrt{x_1^2 + x_2^2} \qquad \tan\theta = \frac{y}{x} \tag{5.5}$$

The range of Cartesian components x_1, x_2 is all real values from $-\infty$ to ∞. The range of r is all positive values and θ takes values between 0 and 2π. A given value of r, and two values of θ differing by 2π and correspond to the same point in the plane.

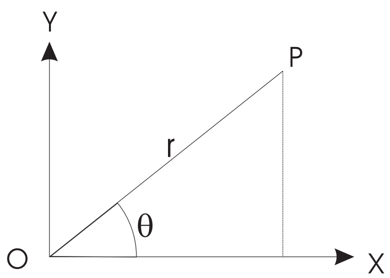

Fig. **7.**

§3 Cylindrical Coordinates

The cylindrical coordinates for position of point particle are denoted by ρ, ϕ, x_3 and represent the values as shown in the figure. Here x_3 is common to the cylindrical and Cartesian systems.

The relation between the Cartesian coordinates x_1, x_2 and cylindrical coordinates ρ, ϕ

$$x_1 = r\cos\phi \qquad x_2 = r\sin\sin\phi \tag{5.6}$$

$$\rho = \sqrt{x_1^2 + x_2^2} \qquad \tan\phi = \frac{y_2}{y_1} \tag{5.7}$$

The above relationship is similar to, except for a change of notation, to given above for plane polar coordinates and the Cartesian coordinates..

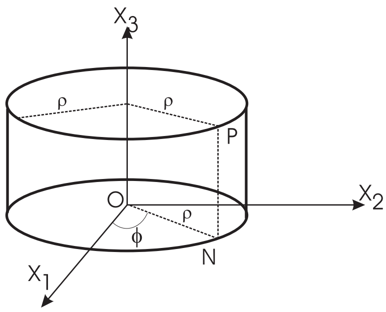

Fig. **8.**

§4 Spherical Polar Coordinates

The **spherical polar coordinates** of a point P are denoted by (r, θ, ϕ). Here r is the distance of the point P from the origin. We have the following relations between r, θ, ϕ and the Cartesian coordinates.

$$x_1 = r\sin\theta\cos\phi \qquad x_2 = r\sin\theta\sin\phi \qquad x_3 = r\cos\theta$$

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2} \qquad \cos\theta = \frac{x_3}{\sqrt{x_1^2+x_2^2}} \qquad \tan\phi = \frac{x_2}{x_1}$$

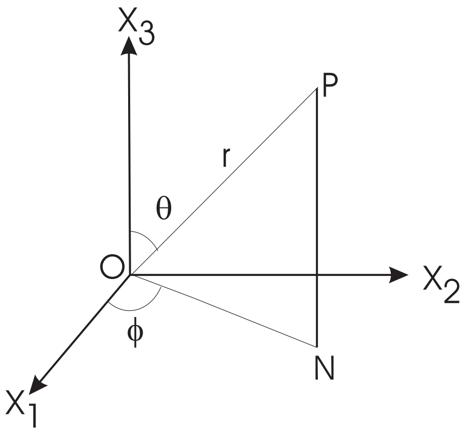

Fig. **9.**

While θ is the angle between the X_3 axis and the line OP, ϕ is the angle between the X_1 axis and the projection of the line OP on the X_1-X_2 plane. It should be noted that ϕ in spherical polar coordinate system is same angle as in the cylindrical coordinate system.

The range of values of r is from 0 to ∞, θ takes values from 0 to π and ϕ has the range 0 to 2π. The points on the positive X_3 axis correspond to $\theta = 0$ and the points on the negative X_3 axis have values $\theta = \pi$. The set of points with fixed values of $r(= R)$ and all values of θ, ϕ correspond to a sphere of radius R with the center at the origin.

§5 Rotation of Coordinate System

Let K and K' be two sets of coordinate axes in a plane with common origin. Fig.10 show two such sets of axes where K' is obtained by rotating the axes K about the origin by an angle α. Let P be a point, as shown in the figure, having coordinates (x_1, x_2) and $(x_1', x_2,')$ w.r.t. the two sets of axes. Then simple geometrical arguments can be used to prove that

$$x_1' = x_1 \cos\alpha + x_2 \sin\alpha \tag{5.8}$$

$$x_2' = -x_1 \sin\alpha + x_2 \cos\alpha \tag{5.9}$$

These equations can be written in matrix form as

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{5.10}$$

The above rotation in a plane about the origin corresponds to a rotation about the x_3 axis in three dimensions, and we can represent it as

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha, & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{5.11}$$

The matrix multiplication gives equation $x_3' = x_3$ in addition to the two equations Eq.(5.8)-

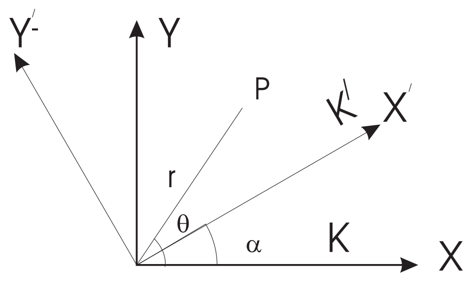

Fig. **10.**

(5.9) as above.

A general rotation in three dimensions can be built up from a succession of rotations about the three coordinate axes. We summarize the results about the rotations about the three coordinate axes in the table given below.

X_1- axis	X_2- axis	X_3- axis
$x_1' = x_1$	$x_2' = x_2$	$x_3' = x_3$
$x_2' = x_2 \cos\alpha + x_3 \sin\alpha$	$x_3' = x_3 \cos\alpha + x_1 \sin\alpha$	$x_1' = x_1 \cos\alpha + x_2 \sin\alpha$
$x_3' = -x_2 \sin\alpha + x_3 \cos\alpha$	$x_1' = -x_3 \sin\alpha + x_1 \cos\alpha$	$x_2' = -x_1 \sin\alpha + x_2 \cos\alpha$

These entries in different columns different are obtained by making *cyclic replacements*, $x_1 \to x_2 \to x_3$, in the first column which is graphically represented .

Lesson 6

A Quick Review of Complex Numbers

[1] Aug 1, 2012

§1 Complex Numbers

It is easy to see that not all polynomials have zero in the real numbers. Simplest example is the polynomial $x^2 + 1$ which cannot become zero for any real x. Complex numbers arose as a result of attempts to make the number system complete in the sense of every polynomial having a root in the number system.

We introduce an imaginary unit as $i \equiv \sqrt{-1}$ with the following rules of multiplication.

$$i^2 = -1, \quad i^3 = -i \quad i^4 = 1; \tag{6.1}$$

A complex number z is represented as a sum of the form $z = x + iy$. For complex number $z = x + iy$, x is called the **real part** and y is called the **imaginary part** of the complex number z. The real and imaginary parts of a complex number z will be denoted by $\Re(z)$ and $\Im(z)$. The rules of addition and multiplication of complex numbers are as follows.

1. Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal if and only if $x_1 = x_2$ and $y_1 = y_2$.
$$z_1 = z_2 \Longleftrightarrow x_1 = x_2, \text{ and } y_1 = y_2 \tag{6.2}$$
2. The sum of two complex numbers z_1, z_2 is given by a complex number
$$z_1 + z_2 = (x_1 + x_2) + i(x_2 + iy_2) \tag{6.3}$$
whose real and imaginary parts are, respectively, the sum of real parts and the imaginary parts of the two given complex numbers.

[1] Lessons 6+7

3. The product of two complex numbers is defined making use of (6.1).

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \end{aligned} \tag{6.4}$$

It is not the need of the hour to introduce complex number system in an axiomatic fashion. So we continue our discussion in an informal manner. All operations carried out with the above rules Eq.(6.1), Eq.(6.4) and the usual rules for real numbers turn out to be correct.

The **complex conjugate** of a complex number z will be denoted by $\bar{z}$. Thus if $z = x + iy$, $\bar{z} = x - iy$ where $x, y \in \mathbb{R}$. The quantity $\sqrt{(x^2 + y^2)}$ will be denoted by $|z|$ and will be called the absolute value, or modulus of complex number z. Note that the quantity

$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2. \tag{6.5}$$

is real and positive for all $z \neq 0$. Division of two complex numbers z_1/z_2, where $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$, can be expressed in the form $a + ib$,with a, b real in the following fashion.

$$\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{z_2\bar{z}_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} \tag{6.6}$$

$$\begin{aligned} &= \frac{(x_1 + iy_1)(x_2 - iy_2)}{x_2^2 + y_2^2} \\ &= \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} \\ &= \frac{(x_1x_2 + y_1y_2)}{x_2^2 + y_2^2} + i\frac{(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} \end{aligned} \tag{6.7}$$

Some simple properties involving the complex conjugate are listed below.

1. $\overline{(\bar{z})} = z$.
2. $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$.
3. $\overline{(z_1 z_2)} = \bar{z}_1\bar{z}_2$.
4. $\overline{(z_1/z_2)} = \bar{z}_1/\bar{z}_2, \qquad z_2 \neq 0$.
5. $(\bar{z})^N = \overline{(z^N)}$.
6. $\Re(z) = \frac{1}{2}(z + \bar{z})$.
7. $\Im(z) = \frac{1}{2i}(z - \bar{z})$.
8. $1/z = \bar{z}/|z|^2\ , \quad z \neq 0$.
9. $|\bar{z}| = |z|$.
10. $|z|^2 = z\bar{z}$.

§2 Polar Form of Complex Numbers

Geometrically, a complex number $z = x + iy$ is represented by a point, with (x, y) as the coordinates, in a two dimensional plane called the **complex plane** or the **Argand diagram**. Let r and θ denote the polar coordinates of a point, z, we have (Fig.11)

$$r = \sqrt{x^2 + y^2} \ , \qquad \theta = \tan^{-1}(y/x). \tag{6.8}$$

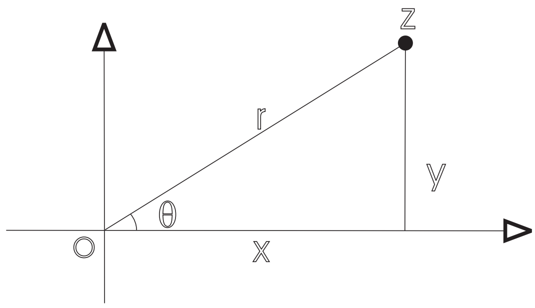

Fig. **11.**

Noting that

$$x = r\cos\theta, \quad y = r\sin\theta \tag{6.9}$$

the complex number z itself can be written as

$$z = r(\cos\theta + i\sin\theta). \tag{6.10}$$

The numbers r and θ are called the **modulus** and the **argument** of the complex number z and will be denoted by $\mod(z)$ and $\arg(z)$ respectively. Note that $\mod(z)$ is just $|z|$ introduced above. Given a non zero complex number z, its argument does not have a unique value; there are infinite values differing by multiples of 2π. For many purposes, it is useful to restrict the allowed range for the polar angle θ so as to get a unique value of θ for a given z. If the value of $\theta = \arg(z)$ is restricted to lie in the range $-\pi < \theta \leq \pi$, we obtain the **principal value** of $\arg(z)$, which will be denoted by $\text{Arg}(z)$, *i.e.*,

$$\text{Arg}(z) = \theta, \qquad -\pi < \theta \leq \pi.$$

It must be noted that $\arg(z)$ is not defined for $z = 0$. In the following we summarize a few properties of the modulus and the argument. Using r_1, r_2, and r_3 denote the moduli and θ_1, θ_2, and θ_3 denote the arguments of the complex numbers z_1, z_2, and z_3, respectively, we have

1. If $z_3 = z_1 z_2$, $r_3 = r_1 r_2$ and $\theta_3 = \theta_1 + \theta_2$.
2. If $z_3 = z_1/z_2, r_3 = r_1/r_2$ and $\theta_3 = \theta_1 - \theta_2, \qquad r_2 \neq 0.$
3. If $z_2 = 1/z_1, r_2 = 1/r_1$ and $\theta_2 = -\theta_1, \qquad r_1 \neq 0.$
4. If $z_2 = z_1^N, r_2 = r_1^N$ and $\theta_2 = N\theta_1$.

The relations between the arguments given above hold modulo 2π.

The exponential of a complex number

We define the exponential of a *complex number* $z = x + iy$, $\exp(z)$ in terms of the exponential, the sine and the cosine functions of a *real variable* as follows:

$$\exp(z) = \exp(x)(\cos y + i \sin y). \tag{6.11}$$

Recalling Euler's theorem

$$\exp(i\theta) = \cos\theta + i\sin\theta \tag{6.12}$$

and noting Eq.(6.10) the complex number z itself can be written as

$$z = r\exp(i\theta) \tag{6.13}$$

in terms of its argument θ and modulus r. This form of complex numbers is called the **polar form**.

The exponential of a complex number, just as that of a real number, is given by a convergent series

$$\exp(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots \tag{6.14}$$

It must be remarked that the sum of an infinite series like the one that appears here has to be defined by means a limiting procedure. You will learn about infinite series in your mathematics courses.

Some important properties of the exponential function are

- $\exp(0) = 1$,
- $\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$
- $\exp(z) \neq 0$ for all z, ($\because \exp(z)\exp(-z) = 1$).

Example 1: We now give some simple examples of manipulating complex numbers.

(1.a). The product of a complex number z with its complex conjugate is always a positive real number because

$$z\bar{z} = (x + iy)(x - iy) = x^2 - i^2y^2 = x^2y^2 \tag{6.15}$$

(1.b). The inverse of a non zero complex number $a+ib$ is easily computed making use of the above property.

$$\frac{1}{z} = \frac{1}{a+ib} \times \frac{a-ib}{a-ib} = \frac{a-ib}{a^2+b^2}. \tag{6.16}$$

Here in the first step we have multiplied and divided by $\bar{z} = a - ib$.

(1.c). A ratio of two complex numbers can be simplified in a similar fashion. For example

$$\frac{2+i}{3i+1} = \frac{2+i}{3i+1} \times \frac{3i-1}{3i-1} = \frac{6i - 2 + 3i^2 - i}{-9-1} = \frac{-5+5i}{-10} = \frac{1}{2}(1-i). \tag{6.17}$$

In the next lecture will use complex numbers to derive a relation between coordinates of a point P in two different set of coordinate axes which are related by a rotation.

Lesson 7

An Application of Complex Numbers

Aug 3, 2012

§1 Introduction

In this lecture we demonstrate usefulness of complex numbers by deriving a relation between coordinates of point in space w.r.t. two sets of coordinate axes related by a rotation about z axis. This result will be useful later when we discuss rotations about an arbitrary axis. The results on rotations will be needed for discussion of equations of motion in rotating frames and for rigid body dynamics,

§2 Rotations about z axis

Using the polar form of complex numbers it is easy to derive a relation between the coordinates of a point when the coordinate axes are rotated about one of the axes.

Let the axes OX', OY' be obtained from OX, OY by performing a rotation about the z- axis by an angle α. Let (x, y, z) and (x', y', z') be the coordinates of a point P w.r.t. the two coordinate systems. Then $z' = z$ and we need only to find a relation between the x, y coordinates of the point in the two sets of axes.

Introducing the polar coordinates (r, θ) and (r', θ) w.r.t. the two system of axes, we see that

$$r' = r \qquad \theta' = \theta - \alpha \tag{7.1}$$

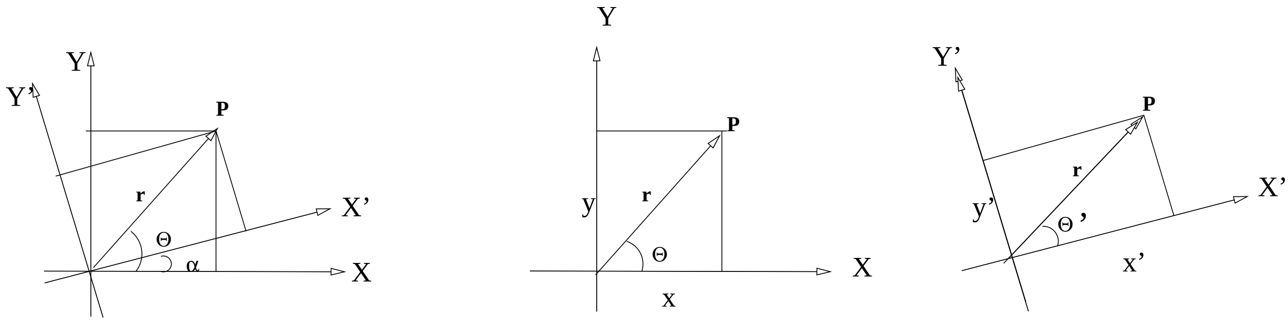

Fig. **12.** Rotation about z axis

Thus, we get

$$\begin{aligned} x' &= r'\cos\theta' = r\cos(\theta-\alpha) \\ &= r\cos\theta\cos\alpha + r\sin\theta\sin\alpha \\ &= x\cos\alpha + y\sin\alpha. \end{aligned} \tag{7.2}$$

Similarly,

$$\begin{aligned} y' &= r'\sin\theta' = r\sin(\theta-\alpha) \\ &= r\sin\theta\cos\alpha - r\cos\theta\sin\alpha \\ &= -x\sin\alpha + y\cos\alpha. \end{aligned} \tag{7.3}$$

When written in a matrix form, these equations become

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{7.4}$$

Similarly, for a rotation about the x- axis we will have $x' = x$ and

$$\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}. \tag{7.5}$$

Finally, for a rotation about the y axis we will have $y' = y$ and

$$\begin{pmatrix} x' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}. \tag{7.6}$$

We use the notation to as

$$R_{\hat{n},\theta} : \vec{x} \overset{(\vec{n},\theta)}{\longrightarrow} \vec{x}' \tag{7.7}$$

indicate the transformation under a rotation by an angle θ about an axis $\vec{n}$.

§3 Group structure of rotations

The set of rotation transformations about a *fixed axis* by all possible angle have the following properties.

- A rotation by zero angles leaves the coordinates unchanged.
- Two rotations one by an angle θ followed by a second rotation by angle ϕ is equivalent to a single rotation by an angle $\theta + \phi$. If $\vec{x} \stackrel{(\vec{n},\theta)}{\longrightarrow} \vec{x}'$ and $\vec{x} \stackrel{(\vec{n},\phi)}{\longrightarrow} \vec{x}''$ then

$$\vec{x} \stackrel{(\vec{n},\theta+\phi)}{\longrightarrow} \vec{x}''$$

- The operation of rotation by an angle $-\theta$ is inverse of rotation by an angle θ.
- Three successive rotations by three different angles can be combined in any way and will give the same result.

  To be precise let R_1, R_2, R_3 denote rotations by angles θ, ϕ, ψ denote angles of three rotations about a fixed axis. Then effect of R_1 following the result of and let R_{12} be the result of R_2 followed by R_1 and R_{23} be the result of R_3 followed by R_2. Then the effect of R_1 after R_{23} is the same as that of R_{12} after R_3. Symbolically,

$$R_{12} = R_1.R_2, \qquad R_{23} = R_2.R_3, \tag{7.8}$$
$$(R_{12}).R_3 \quad = \quad R_1(R_{23}). \tag{7.9}$$

  The above properties make the set of all rotations about a fixed axis a *group*. You will learn more about groups in courses in later semesters.

  The above four group properties are obvious for rotations about a fixed axis. That the set of all possible rotations $(\hat{n}, \theta)$, by all possible angles θ and about all possible axes $\hat{n}$ passing through the origin, also form a group, is not so obvious.

  We shall study up the rotation group in later lectures in this course.

Complex Numbers and Rotations

The transformation under a rotation can described in terms of multiplication of complex numbers. To this end we consider a rotation by angle α about the z axis. Let $\xi = x + iy$ and let $\xi' = x' + y'$ be the coordinates w.r.t. the rotated axes. Then the relation

$$\xi' = \exp(-i\alpha)\xi = (\cos\alpha - i\sin\alpha)\xi \tag{7.10}$$

correctly describes the effect of rotations on the coordinates (x, y). This is most easily seen by writing $\xi = (x+iy)$ in the right hand side, multiplying and comparing the real and imaginary parts of the two sides of the above equation. We will verify that this process indeed laeds to the correct relation as described above. Consider the right hand side of the above equation

$$\begin{aligned} \text{R.H.S.} &= (\cos\alpha - i\sin\alpha)(x+iy) && (7.11)\\ &= \cos\alpha\, x - i\cos\alpha\, y + i\sin\alpha\, x + \sin\alpha\, y) && (7.12)\\ &= x\cos\alpha + y\sin\alpha - ix\sin\alpha x + iy\cos\alpha). && (7.13) \end{aligned}$$

The real and imaginary parts when equated to x' and y' give us the desired result

$$\begin{aligned} x' &= x\cos\alpha + y\sin\alpha, && (7.14)\\ y' &= -x\sin\alpha + y\cos\alpha. && (7.15) \end{aligned}$$

Lesson 8

Rotation of Coordinate Systems-I

Aug 6, 2012

Today I will derive relation between components of a vector w.r.t. to two sets of axes which are related by a rotation. Let me remind you of my notation: I have two sets of axes K and K'. Here K' denotes a set of axes obtained by rotating the axes K about an axis passing through the origin and has the direction along unit vector $\hat{n}$.

This part of the material is not available to you anywhere. I will be putting it up on the internet, but that will take some time. So I suggest that you take down the notes carefully.

Why am I teaching this topic? Let me give you some idea. This will be useful in several places. For example one application will be in discussion of motion in a rotating frame. The other topic, where results derived today will be needed, will be rigid body dynamics.

I hope to give you some flavour of rigid body dynamics beyond what is normally covered in the 12^{th} class and beyond what is presented in text books at B.Sc. level. Normally what you get in the first treatment is a discussion of rigid body motion only in special cases where the axis of rotation is fixed. I hope to go beyond this and cover the most general case of rigid body dynamics.

§1 Two "Pictures" of Rotations

So let me begin with drawing a picture. We have a set of axes X_1, X_2, X_3 and X_1', X_2', X_3' related by rotation by an angle α about the axis $\hat{n}$. I will remind you of an important property that we discussed last time. This will be helpful in visualising some results intuitively, rather than deriving through a sequence of mathematical steps.

Let us recall that a rotation can be viewed in two different ways: (a) rotation of coordinate axes and this is the view that will be normally used by me in my lectures; (b) rotation of vectors.

In the first *picture* we consider coordinates of a point P in the two sets of axes K and K' related by a rotation given by $(\hat{n}, \alpha)$. We consider a point P in space and its coordinates w.r.t. the axes K will be denoted by (x_1, x_2, x_3). The same point P will have different coordinates (x_1', x_2', x_3') w.r.t. the rotated axes K'. we collectively denote of these two sets of values of components by the notation

$$\vec{x} = (x_1, x_2, x_3), \qquad \vec{x}' = (x_1', x_2', x_3'. \tag{8.1}$$

When viewed differently, these equations can also be can be seen as specifying the components of the position vector $\overrightarrow{OP}$, the axes have changed so the components are different.

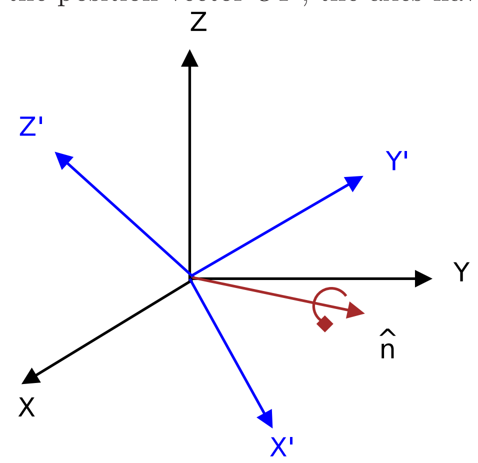

Fig. **13.** Rotating Axes

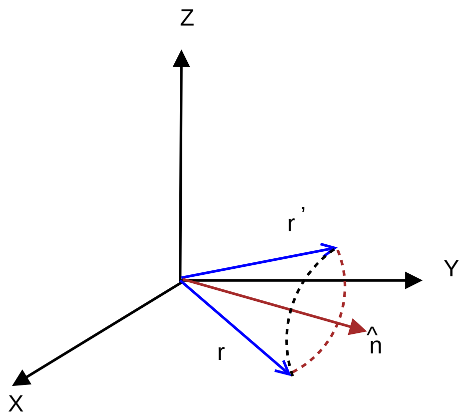

Fig. **14.** Rotate Vectors

In the second *picture* of rotations, we have *only one set of axes* and we rotate the vectors. The figure below shows a set of axes and a vector $\overrightarrow{OP}$ and now rotate the vector about the given axis along the unit vector $\hat{n}$ by some angle, but the rotation is to be performed in the clockwise direction. The point P will move in a circular path as shown in the figure below. The rotation takes the vector $\overrightarrow{OP}$ to a new vector $\overrightarrow{OQ}$ as shown in the figure. Now the vector has changed and we can ask for relation between the components of the two vectors w.r.t. a given fixed set of axes.

We now have two points P and Q related by a rotation. We can now ask for relation between the coordinates of the two points P, Q w.r.t. *the same set of axes* and we write

$$P : (x_1, x_2, x_3), \qquad Q : (x_1', x_2', x_3').. \tag{8.2}$$

The two ways of performing rotations are related. The result that is almost obvious is that effect of an anticlockwise rotation of axes by an angle α (first view) will be related to the effect of clockwise rotation of the vector by the same angle α in the second view. Thus, relation between coordinates in Eq.(8.1) and Eq.(8.2) is the same as if the rotation in the second view is performed by the same angle α, as in the first case, but is taken in opposite sense.

§2 Relating Components of two vectors by rotation

Using the second picture we can get the results on rotation quickly in a transparent manner which is easy to visualize. We will use the second picture of rotations as it is more intuitive and easy to visualise. The first picture will also give results but the derivation will be more mathematical in nature because it is not easy to visualise a similar change in the first picture when the axe are rotated.

The top view of the effect of rotation on the point P is shown in figure below. The point P moves on a circular path, with centre shown as N, by an angle α as shown and reaches a new position Q. The point N is a point on the axis of rotation.

§3 Infinitesimal Rotations

Form the last lecture we know that the effect of two successive rotations by angles α and β is same as the effect of a single rotation by angle $\alpha+\beta$. This tells us that any rotation, about a given axis, can be thought of as being a result of a large number of rotations by small angle about the same axis. For this reason it turns out to be sufficient to consider rotations by small angles, infinitesimal angles only. This is very helpful in getting our answers quickly in a transparent manner.

Remembering that the plane of circle in Fig.**??** will be perpendicular to the axis of rotation, the difference in the position vectors of the point P and Q, $\overrightarrow{PQ} \equiv \overrightarrow{OP} - \overrightarrow{OQ}$, is given by $\overrightarrow{NQ}$ and is perpendicular to the axis of rotation.

It will now be proved that also for *infinitesimal rotations* the change in vector, given by $\overrightarrow{PQ}$, will also be perpendicular to the $\overrightarrow{NQ}$. Writing $\vec{x}' = \vec{x} + \overrightarrow{\Delta x}$, we therefore have

$$\begin{aligned} \hat{n} \cdot \overrightarrow{\Delta x} &= 0, && (8.3)\\ \text{and} \quad \vec{x} \cdot \overrightarrow{\Delta x} &\approx 0 \quad \text{for } \alpha \approx 0. && (8.4) \end{aligned}$$

Alternatively, the above results can also be seen as follows. The distance of the point P and Q from the origin will be equal. Hence

$$\begin{aligned} \vec{x} \cdot \vec{x} &= \vec{x}' \cdot \vec{x}' \\ &= (\vec{x} + \overrightarrow{\Delta x}) \cdot (\vec{x} + \overrightarrow{\Delta x}) \\ &= \vec{x} \cdot \vec{x} + 2\vec{x} \cdot \overrightarrow{\Delta x} + \overrightarrow{\Delta x} \cdot \overrightarrow{\Delta x} \\ &\approx \vec{x} \cdot \vec{x} + 2\vec{x} \cdot \overrightarrow{\Delta x}. \qquad (8.5) \end{aligned}$$

In the last step the quantity $\overrightarrow{\Delta x} \cdot \overrightarrow{\Delta x}$ has been neglected as it will be small compared to the other two terms for infinitesimal rotations. Therefore Eq.(8.5) leads to the desired result Eq.(8.4). Similarly, both the vectors $\overrightarrow{NP}$ and $\overrightarrow{NQ}$ lie in a plane perpendicular to the axis of rotation. Since $\overrightarrow{OQ} - \overrightarrow{OP} = \overrightarrow{NP} - \overrightarrow{NP}$

$$\begin{aligned} \overrightarrow{\Delta x} \cdot \hat{n} &= (\vec{x}' - \vec{x}) \cdot \hat{n} \\ &= (\overrightarrow{OQ} - \overrightarrow{OP}) \cdot \hat{n} \\ &= (\overrightarrow{NQ} - \overrightarrow{NP}) \cdot \hat{n} \\ &= \overrightarrow{PQ}. \cdot \hat{n} \\ \therefore \quad \overrightarrow{\Delta x} \cdot \hat{n} &= 0. \qquad (8.6) \end{aligned}$$

The last step follow from the fact that the circular arc PQ lies in the plane perpendicular to the axis. Thus, once again, we get the result that the difference $\overrightarrow{\Delta x} = \vec{x}' - \vec{x}$ is always perpendicular to $\hat{n}$. For a proof of this result it was not necessary to take the angle of rotation to be small.

Thus we see that the vector $\overrightarrow{\Delta x}$ is perpendicular to both vectors $\vec{x}$ and $\hat{n}$ and hence it is along the direction of the cross product $\hat{n} \times \vec{x}$, or opposite to the cross product. Thus in both the cases we can write

$$\begin{aligned} \overrightarrow{\Delta x} &\propto \alpha(\hat{n} \times \vec{x}) \\ &= \lambda\alpha(\hat{n} \times \vec{x}), \end{aligned} \tag{8.7}$$

where λ is a proportionality constant to be fixed. It will be positive in case $\overrightarrow{\Delta x}$ is parallel to $\hat{n}$, or negative otherwise. The angle α has been put explicitly in the above relation, because when the angle of rotation is zero, the difference $\overrightarrow{\Delta x}$ will be zero.

We have obtained the result for change in components of a vector under a small rotation. Let us now ask what happens if angular rotation is not small? How do the vector components change? That is our final objective.

§4 Finite Rotations

What about finite rotations? That is our final aim, *i.e.* to derive a relation between $\vec{x}'$ and $\vec{x}$ when the angle of rotation is not small.

Note that the difference $\overrightarrow{\Delta x} = \vec{x}' - \vec{x}$ is vector $\overrightarrow{PQ}$ which lies in plane perpendicular to the vector $\vec{n}$. We want to represent this vector as a sum of two vectors in the plane. I want to get hold of two vectors such that the difference $\overrightarrow{\Delta x}$ can be resolved in terms of the two vectors. All that we need is that the two vectors should not be collinear.

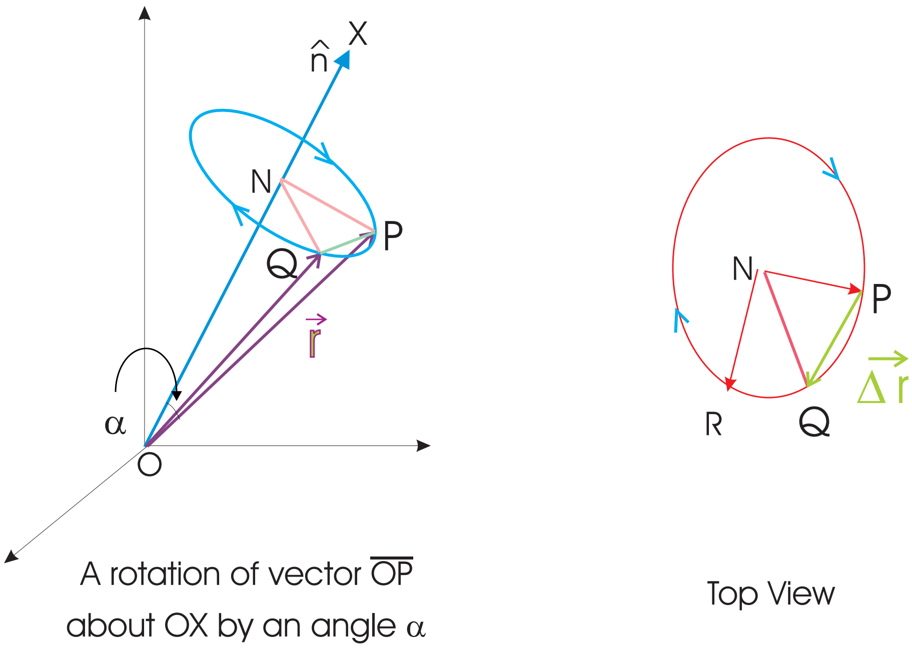

Fig. **15.** Rotation of a Vector about an axis

Let us now assume that the point P is not on the axis of rotation and does not coincide with the origin. In this case $\vec{x}$ is not parallel to the axis of rotation. You can see from the figure that the vector $\hat{n} \times \vec{x}$ is along the tangent to the circle at P and $\hat{n} \times (\hat{n} \times \vec{x})$ is towards the centre of the circle at the point P. Therefore, vectors $(\hat{n} \times \vec{x}) \equiv \vec{u}$ and $\hat{n} \times (\hat{n} \times \vec{x}) \equiv \vec{v}$ are non zero perpendicular vectors lying in the plane perpendicular to the axis.

Every vector in the plane $\perp$ to the axis can, therefore, be resolved along these two vectors $\vec{u}, \vec{v}$. Hence we can write the difference $\overrightarrow{\Delta x}$ as a combination of vectors $\vec{u}$ and $\vec{v}$ with some coefficients A and B. This gives

$$\begin{aligned} \overrightarrow{\Delta x} &= A\vec{u} + B\vec{v} \\ &= A(\hat{n} \times \vec{x}) + B\hat{n} \times (\hat{n} \times \vec{x}) \qquad (8.8) \\ \text{and} \quad \vec{x}' &= \vec{x} + A(\hat{n} \times \vec{x}) + B\hat{n} \times (\hat{n} \times \vec{x}) \qquad (8.9) \end{aligned}$$

We make a slight change in our notation. The angle of rotation will be denoted by θ and the dependence of the final position vector on θ will be shown explicitly by writing $\vec{x}' = \vec{x}'(\theta)$. The vector $\vec{x}$ is just the value of $\vec{x}'(\theta)$ for $\theta = 0$.

Here the coefficients will, of course depend of the angle of rotation θ. Therefore we write

$$\begin{aligned} \overrightarrow{\Delta x}(\theta) &= A(\theta)(\hat{n} \times \vec{x}) + B(\theta)\hat{n} \times (\hat{n} \times \vec{x}) \qquad (8.10) \\ \text{and} \quad \vec{x}'(\theta) &= \vec{x} + A(\theta)\,(\hat{n} \times \vec{x}) + B(\theta)\,\hat{n} \times (\hat{n} \times \vec{x}) \qquad (8.11) \end{aligned}$$

Here $\vec{x}$ denotes components of the initial vector.

Now our job is to fix $A(\theta)$ and $B(\theta)$. How do we go about fixing them? There will be several ways of doing it. One way is to use matrices. Since some of you have not had enough exposure ot matrices, I will avoid using matrices. I will use only vector algebra and some amount of calculus and ordinary differential equations. All this is within your reach and I will provide all details where ever needed. Please feel free to ask questions if you are not able to follow.

It may be little long process, but is fairly simple and straightforward. Let me at first outline the steps. I want to differentiate the vector $\vec{x}'(\theta)$ w.r.t. θ and derive differential equations for A and B which will be solved to get the final answer.

So let me remind you of the derivative of function of real variable $f(x)$. If I have a function $f(x)$, of a real variable x, it derivative is defined as the limit

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

§5 Calculate the derivative of $\vec{x}'(\theta)$ w.r.t. θ

If we apply a rotation by small angle $\Delta\theta$, the components $\vec{x}'(\theta)$ of the vector will change by a amount given by Eq.(8.6):

$$\begin{aligned} \vec{x}(\theta+\Delta\theta) - \vec{x}'(\theta) &= \Delta\theta\lambda(\hat{n}\times\vec{x}'(\theta)). && (8.12) \\ \Rightarrow \qquad \frac{\vec{x}(\theta+\Delta\theta) - \vec{x}'(\theta)}{\Delta\theta} &= \lambda(\hat{n}\times\vec{x}'(\theta)). && (8.13) \end{aligned}$$

Therefore, taking limit $\Delta\theta \to 0$ gives

$$\frac{d\vec{x}'(\theta)}{d\theta} = \lambda(\hat{n}\times\vec{x}'(\theta)). \tag{8.14}$$

If you accept this, the rest is fairly straight forward algebra.

§6 Derive differential equations for A and B

Next, using Eq.(8.11) in the left hand side of Eq.(8.14) gives

$$\begin{aligned} \text{L.H.S. of (8.14)} &= \frac{d\vec{x}'(\theta)}{d\theta} \\ &= 0 + \frac{dA(\theta)}{d\theta}(\hat{n}\times\vec{x}) + \frac{dB(\theta)}{d\theta}\big\{\hat{n}\times(\hat{n}\times\vec{x})\big\} && (8.15) \end{aligned}$$

and substituting for $x(theta)$ from Eq.(8.11) the right hand side of Eq.(8.14) becomes

$$\begin{aligned} \text{R.H.S.} &= \lambda\big[(\hat{n}\times\vec{x}) + \hat{n}\times\big\{A(\theta)(\hat{n}\times\vec{x}) + B(\theta)\hat{n}\times(\hat{n}\times\vec{x}\big\}\big] \\ &= \lambda\big[(\hat{n}\times\vec{x}) + A(\theta)\hat{n}\times(\hat{n}\times\vec{x}) + B(\theta)\hat{n}\times\big\{\hat{n}\times(\hat{n}\times\vec{x})\big\}\big]. && (8.16) \end{aligned}$$

The vector identity $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$ will now be used repeatedly to simplify the last expression giving

$$\begin{aligned} \hat{n} \times (\hat{n} \times \vec{x}) &= ((\hat{n} \cdot \vec{x})\hat{n} - \vec{x}) && (8.17) \\ \hat{n} \times \big\{\hat{n} \times (\hat{n} \times \vec{x})\big\} &= \hat{n} \times \big\{(\hat{n} \cdot \vec{x})\hat{n} - (\hat{n} \cdot \hat{n})\vec{x}\big\} \\ &= (\hat{n} \times \hat{n})(\hat{n} \cdot \vec{x}) - (\hat{n} \cdot \hat{n})(\hat{n} \times \vec{x}) \\ &= = 0 - \hat{n} \times \vec{x} = -\hat{n} \times \vec{x}. && (8.18) \end{aligned}$$

The properties $\hat{n} \cdot \hat{n} = 1$ and $\hat{n} \times \hat{n} = 0$ have been used to simplify the expression in the second step. Substituting (8.18) in (8.16) we get

$$\begin{aligned} \text{R.H.S. of (8.14)} &= \lambda\big[\vec{x} + A(\theta)\hat{n} \times (\hat{n} \times \vec{x}) + B(\theta)\hat{n} \times \big\{\hat{n} \times (\hat{n} \times \vec{x})\big\}\big] \\ &= \lambda\vec{x} + \lambda A(\theta)\hat{n} \times (\hat{n} \times \vec{x}) - \lambda B(\theta)\hat{n} \times \vec{x} && (8.19) \\ &= \big(\lambda - \lambda B(\theta)\big)(\hat{n} \times \vec{x}) + \lambda A(\theta)\hat{n} \times (\hat{n} \times \vec{x}) && (8.20) \end{aligned}$$

Equating expressions in Eq.(8.15) and (8.20), we get

$$\begin{aligned} &\frac{dA(\theta)}{d\theta}(\hat{n} \times \vec{x}) + \frac{dB(\theta)}{d\theta}\big\{\hat{n} \times (\hat{n} \times \vec{x})\big\} && (8.21) \\ &= \big(\lambda - \lambda B(\theta)\big)(\hat{n} \times \vec{x}) + \lambda A(\theta)\hat{n} \times (\hat{n} \times \vec{x}) && (8.22) \end{aligned}$$

We can now compare coefficients of the linearly independent vectors $\hat{n}\times\vec{x}$ and $\hat{n}\times(\hat{n}\times\vec{x})$ to get

$$\begin{aligned} \frac{dA(\theta)}{d\theta} &= \lambda - \lambda B(\theta) && (8.23) \\ \text{and} \quad \frac{dB(\theta)}{d\theta} &= \lambda A(\theta) && (8.24) \end{aligned}$$

We will continue from here and in the next lecture these differential equations will be solved. You may ask questions on material covered so far.

Lesson 9

Rotation of Coordinate System-II

Aug 6, 2012

In the morning session we have started constructing a proof of the basic formula for rotation of vectors.

$$\vec{x}\,'(\theta) = \vec{x} + \sin\theta(\hat{n}\times\vec{x}) + (1-\cos\theta)\hat{n}\times(\hat{n}\times\vec{x}). \tag{9.1}$$

Here $\vec{x}$ gives the position vector of a point P, $\vec{x}\,'(\theta)$ is the position vector of the same point after rotations of coordinate axes by an angle θ with the axis of rotation specified by the unit vector $\hat{n}$.

It has been already shown that $\vec{x}\,'(\theta)$ has the form

$$\vec{x}\,'(\theta) = \vec{x} + A(\theta)(\hat{n}\times\vec{x}) + B(\theta)\hat{n}\times(\hat{n}\times\vec{x}). \tag{9.2}$$

where the functions $A(\theta)$ and $B(\theta)$ satisfy the differential equations

$$\frac{dA(\theta)}{d\theta} = \lambda - \lambda B(\theta), \tag{9.3}$$

$$\text{and}\quad \frac{dB(\theta)}{d\theta} = \lambda A(\theta). \tag{9.4}$$

Note that for $\theta = 0$, $\vec{x}\,'(\theta)$ must reduce to $\vec{x}$, hence we have the initial conditions

$$A(0) = 0 \qquad B(0) = 0. \tag{9.5}$$

To solve these equations let us introduce a function $f(\theta)$ defined by

$$f(\theta) = A(\theta) + i(B(\theta) - 1), \tag{9.6}$$

which satisfies

$$f(\theta)\Big|_{\theta=0} = -i \tag{9.7}$$

Using Eq.(9.3) and (9.4) we derive a differential equation satisfied by $f(\theta)$:

$$\begin{aligned}
\frac{d}{d\theta}f(\theta) &= \frac{dA(\theta)}{d\theta} + i\frac{dB(\theta)}{d\theta} \\
&= \lambda - \lambda B(\theta) + i\lambda A(\theta) \\
&= i\lambda A(\theta) - \lambda(B(\theta) - 1) \\
&= i\lambda(A(\theta) + i(B(\theta) - 1)) \qquad (9.8)\\
\text{or} \qquad \frac{d}{d\theta}f(\theta) &= i\lambda f(\theta) \qquad (9.9)
\end{aligned}$$

This differential equation for $f(\theta)$ has solution

$$f(\theta) = Ne^{i\lambda\theta}, \qquad (9.10)$$

where N is a constant of integration which is easily determined using Eq.(9.7) and we get

$$f(0) = N = -i \implies f(\theta) = -ie^{i\lambda\theta}. \qquad (9.11)$$

Using Euler's theorem $e^{i\lambda\theta} = \cos(\lambda\theta) + i(\sin\lambda\theta)$ in Eq.(9.6) and comparing real and imaginary parts of both sides in the resulting equation

$$-i[\cos(\lambda\theta) + i\sin(\lambda\theta)] = A(\theta) + i(B(\theta) - 1). \qquad (9.12)$$

we get

$$A(\theta) = \sin(\lambda\theta), \quad B(\theta) = 1 - \cos(\lambda\theta). \qquad (9.13)$$

Hence we get the desired answer for rotation of **vectors** by an angle θ about an axis specified by unit vector $\hat{n}$.

$$\vec{x}\,'(\theta) = \vec{x} + \sin(\lambda\theta)(\hat{n}\times\vec{x}) + (1-\cos\lambda\theta)\big(\hat{n}\times(\hat{n}\times\vec{x})\big). \qquad (9.14)$$

If the coordinate axes are rotated, instead of vectors then the formula for relating the components of position vector of a point w.r.t. the two sets of axes is to be obtained by replacement $\theta \to -\theta$ in the above result Eq.(9.14) and we have

$$\vec{x}\,'(\theta) = \vec{x} - \sin(\lambda\theta)(\hat{n}\times\vec{x}) + (1-\cos\lambda\theta)\big(\hat{n}\times(\hat{n}\times\vec{x})\big). \qquad (9.15)$$

Fixing the constant λ

The unknown constant λ will now be fixed by comparing the above answer with the known answer for special case of a rotation about the third axis by taking $\hat{n} = (0,0,1)$. For the special case of rotations about the X_3 axis $\hat{n} = (0,0,1)$ and we compute $(\hat{n}\times\vec{x})$ and $\hat{n}\times(\hat{n}\times\vec{x})$. First we compute $(\hat{n}\times\vec{x})$ using the determinant formula for the cross product.

$$\hat{n}\times\vec{x} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ 0 & 0 & 1 \\ x_1 & x_2 & x_3 \end{vmatrix} = -\hat{e}_1 x_2 + \hat{e}_2 x_1. \qquad (9.16)$$

Next we use this answer to compute $\hat{n} \times (\hat{n} \times \vec{x})$:

$$\hat{n} \times (\hat{n} \times \vec{x}) = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ 0 & 0 & 1 \\ -x_2 & x_1 & 0 \end{vmatrix} = -\hat{e}_1 x_1 - \hat{e}_2 x_2. \tag{9.17}$$

Thus Eq.(9.15)-(9.17), for the special case of a rotation about the X_3 axis, we get

$$\begin{aligned} \vec{x}\,'(\theta) &= \vec{x} - \sin(\lambda\theta)(-\hat{e}_1 x_2 + \hat{e}_2 x_1) + (\cos(\lambda\theta) - 1)(\hat{e}_1 x_1 + \hat{e}_2 x_2) & (9.18) \\ &= [x_1 \cos(\lambda\theta) + x_2 \sin(\lambda\theta)]\hat{e}_1 + [-x_1 \sin(\lambda\theta) + x_2 \cos(\lambda\theta)]\hat{e}_2 + x_3 \hat{e}_3. & (9.19) \end{aligned}$$

These equations give

$$\begin{aligned} x_1'(\theta) &= x_1 \cos(\lambda\theta) + x_2 \sin(\lambda\theta), & (9.20) \\ x_2'(\theta) &= -x_2 \sin(\lambda\theta) + x_2 \cos(\lambda\theta), & (9.21) \\ x_3'(\theta) &= x_3. & (9.22) \end{aligned}$$

These equation when compared with the known transformation equations for the components of the position vector of a point for a rotation about the third axis

$$\begin{aligned} x_1'(\theta) &= x_1 \cos\theta + x_2 \sin\theta, & (9.23) \\ x_2'(\theta) &= -x_2 \sin\theta + x_2 \cos\theta, & (9.24) \\ x_3'(\theta) &= x_3. & (9.25) \end{aligned}$$

we get $\lambda = 1$. Hence the final rule, for transformation of components of position vector under **rotation of axes**, is given by

$$\boxed{\vec{x}\,'(\theta) = \vec{x} - \sin\theta(\hat{n} \times \vec{x}) + (1 - \cos\theta)\big(\hat{n} \times (\hat{n} \times \vec{x})\big)}. \tag{9.26}$$

Added in Notes : Geometrical Interpretation

You can easily derive the result by geometrical arguments. Some steps are outlined here.

The rotation of a vector $\overrightarrow{OP}$ about an axis OX gives vector $\overrightarrow{OQ}$. The difference in $\overrightarrow{PQ}$.

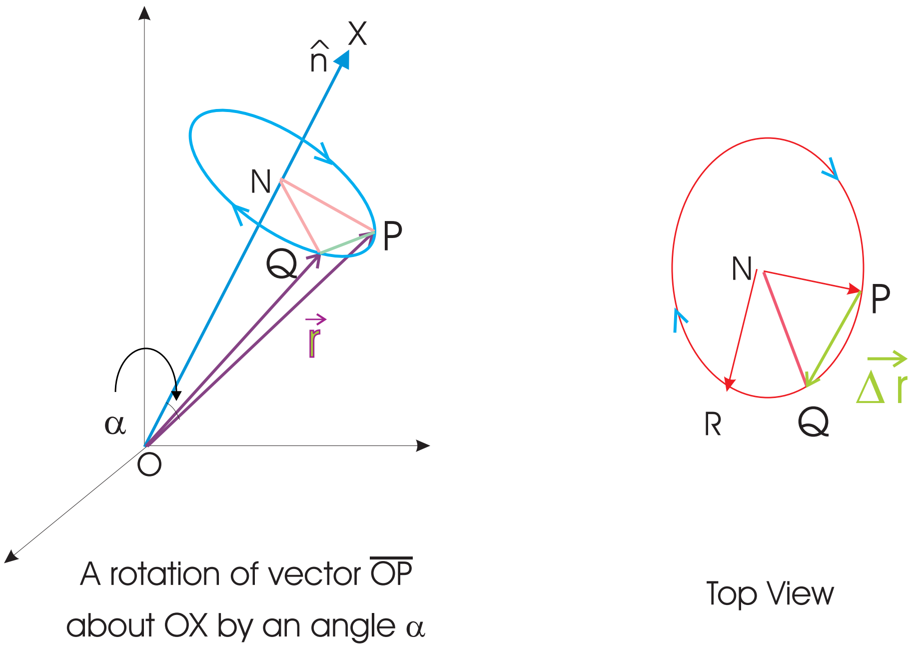

Fig. **16.** Rotation about an axis

Note that we have

$$\overrightarrow{ON} = (\hat{n}\cdot\vec{x})\hat{n} \tag{9.27}$$

$$\overrightarrow{NP} = \overrightarrow{OP} - \overrightarrow{ON} = \vec{x} - (\hat{n}\cdot\vec{x})\hat{n} \tag{9.28}$$

$$\vec{x}\,' = \overrightarrow{ON} + \overrightarrow{NQ}. \tag{9.29}$$

We can resolve the vector $\overrightarrow{NQ}$ along two perpendicular directions of $\overrightarrow{NP}$ and $\overrightarrow{NR}$. Note that the vectors $\hat{n}\times(\hat{n}\times\vec{x})$ and $\hat{n}\times\vec{x}$ are along (either parallel or anti-parallel to the vectors $\overrightarrow{NP}$ and $\overrightarrow{NR}$. Filling the details and completion of the proof of the result Eq.(9.26) is left as an exercise for you.

Lesson 10

Learning Basics of Differentiation

Aug 10, 2012

§1 Prerequisites

It is assumed that you are familiar with sets, operations on sets such as union and intersection etc., and also with functions, limits of sequences, limit and continuity of functions.

§2 Functions

Let X, Y be two sets. A **function** from set X to set Y is a rule which associates with every element $x \in X$ a unique $y \in Y$. It must be emphasized that in order that a function qualifies as well defined on X.

odot the function must be defined for all $x \in X$, and

odot the value of the function must be must be unique for all $x \in X$ and must be a member of the set Y.

Most of the function needed in this course will be functions from the set of all real numbers $\mathbb{R}$ to real numbers $\mathbb{R}$.

Example 2: We will give several examples of functions.

§3 Limits

It is assumed that you are familiar with limits of sequences and limit and continuity of functions. A few important properties of limits are listed below. Let a_n, b_n be sequences such that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then

- $\odot$ $\lim\limits_{n\to\infty} (a_n\, b_n) = a\, b$

- $\lim_{n\to\infty}(a_n + b_n) = a + b$
- $\lim_{n\to\infty} \alpha a_n = \alpha \qquad \alpha \in \mathbb{R}$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b} \qquad \text{if } b \neq 0$

Similar results hold for limits of functions. We list some of these results here. Let $f(x), g(x)$.. be functions which are continuous at $x = x_0$. Then

- $\lim_{x\to x_0} \alpha\, f(x) = \lim_{x\to x_0} \alpha\, f(x)$
- $\lim_{x\to x_0} (f(x) + g(x)) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x)$
- $\lim_{x\to x_0} (f(x)\, g(x)) = \lim_{x\to x_0} f(x)\, \lim_{x\to x_0} g(x)$
- $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} f(x) \Big/ \lim_{x\to x_0} g(x), \qquad \text{if } \lim_{x\to x_0} g(x) \neq 0$

§4 Derivative of a function

Definition 1. *If the limit*

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \tag{10.1}$$

exists, its value is defined to be the differential coefficient of function $f(x)$ and is denoted by $\frac{df}{dx}$. Alternative, shorter, notation $f'(x)$ is also used.

Example 3: We give a few simple examples of derivatives of function.

(3.a). If a function is constant, c, for all x, then $f(x) = c$ and also $f(x+h) = c$. We therefore get

$$\frac{d}{dx} c = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} \frac{0}{h} = 0. \tag{10.2}$$

(3.b). For $f(x) = x$, we have $f(x+h) = x + h$ and

$$\frac{df(x)}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{h}{h}, \tag{10.3}$$

$$\therefore \frac{d}{dx} x = 1. \tag{10.4}$$

(3.c). Let $f(x) = \frac{1}{x}$, then

$$\frac{df(x)}{dx} = \lim_{h\to 0} \frac{1}{h}\Big(\frac{1}{x+h} - \frac{1}{x}\Big) = \lim_{h\to 0} \frac{-1}{x(x+h)}, \tag{10.5}$$

$$\therefore \frac{d}{dx} \frac{1}{x} = -\frac{1}{x^2}. \tag{10.6}$$

(3.d). We compute the derivative of product of two functions $f(x)$ and $g(x)$.

$$\frac{d}{dx}\big(f(x)\,g(x)\big) = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \tag{10.7}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h} \tag{10.8}$$

where we have added and subtracted a term $f(x)g(x+h)$ in the numerator. On rearranging the last expression we get

$$\frac{d}{dx}\big(f(x)\,g(x)\big) = \lim_{h\to 0}\frac{\Big(f(x+h)-f(x)\Big)g(x+h)}{h} + \lim_{h\to 0}\frac{f(x)\Big(g(x+h)-g(x)\Big)}{h} \tag{10.9}$$

$$= \left(\frac{df(x)}{dx}\right) g(x) + f(x)\left(\frac{dg(x)}{dx}\right) \tag{10.10}$$

provided the derivatives in the right hand side exist. This proves the rule of differentiating product of two functions. Other rules of differentiation, listed below, are proved in a similar fashion.

§5 Rules of Differentiation

Let $f(x), g(x), ...$ be functions which are differentiable at x and $\alpha, \beta, ...$ be constants. Then we have

1. $\dfrac{d}{dx}(\alpha f(x)) = \alpha\dfrac{df(x)}{dx} \qquad \alpha \in \mathbb{R}$
2. $\dfrac{d}{dx}(f(x)+g(x)) = \dfrac{df(x)}{dx} + \dfrac{dg(x)}{dx}$
3. $\dfrac{d}{dx}(f(x)\cdot g(x)) = g(x)\,\dfrac{df(x)}{dx} + f(x)\cdot\dfrac{d}{dx}g(x)$
4. $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{f'(x)g(x)-g'(x)f(x)}{(g(x))^2}$, if $g(x) \neq 0$.
5. As a special case of the last result, taking $f(x)=1$, we get $\dfrac{d}{dx}\left(\dfrac{1}{g(x)}\right) = -\dfrac{g'(x)}{(g(x))^2}$, if $g(x)\neq 0$.

Chain Rule

The chain rule gives derivative of a function of a function $f(g(x))$. We use the notation $t \equiv g(x)$ and assume that $f(x), g(x)$ are two functions of a real variable such that the $g'(x)$ exist at x and $f'(t)$ exist at $t = g(x)$, then

$$\frac{d}{dx}\,f(g(x)) = \left(\frac{df(t)}{dt}\right)\Big|_{t=g(x)}\left(\frac{dg(x)}{dx}\right) \tag{10.11}$$

§6 Examples

Example 4: We give a few examples of working of chain rule.

(4.a). We will show that $\frac{d}{dx}\sin x^2 = 2x\cos x^2$. Set $t = x^2$ and compute

$$\begin{aligned} \frac{d}{dx}\sin x^2 &= \frac{d}{dx}\sin t = \left(\frac{d\,\sin t}{dt}\right)\Big|_{t=x^2}\left(\frac{dt}{dx}\right) && (10.12)\\ &= (\cos t)\big|_{t=x^2}\left(\frac{d\,x^2}{dx}\right) = \cos x^2\,(2x) && (10.13)\\ &= 2x\cos x^2 && (10.14) \end{aligned}$$

(4.b). Next we compute the derivative of $\sqrt{x^2+a^2}$, where a is a constant. Let $t(x) = x^2 + a^2$, then

$$\begin{aligned} \frac{d}{dx}\sqrt{x^2+a^2} &= \frac{d\sqrt{t}}{dx} = \left(\frac{d\sqrt{t}}{dt}\right)\left(\frac{dt}{dx}\right) && (10.15)\\ &= \left(\frac{1}{2}t^{-1/2}\right)\Big|_{t=x^2+a^2}\left(\frac{d\,(x^2+a^2)}{dx}\right) = \frac{1}{2}(x^2+a^2)^{-1/2}\,2x && (10.16)\\ &= \frac{x}{x^2+a^2} && (10.17) \end{aligned}$$

Using the chain rule requires some practice, you may take up more exercises from the book by Thomas and Finney.

§7 Geometrical Interpretaton

The derivative of a function $f(x)$ at a point P has a useful geometric interpretation. We plot the curve $y = f(x)$ and draw a tangent AB at the point. The slope of the tangent at the point is equal to the value of the derivative of the function

$$\frac{df(x)}{dx} = \tan\alpha = \frac{y_2 - y_1}{x_2 - x_1} \qquad (10.18)$$

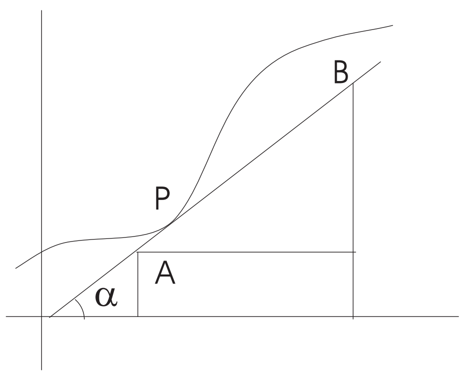

Fig. **17.** Geometrical Interpretation of the Derivative

where (x_1, y_1) and (x_2, y_2) are the coordinates of the two points A and B.

Lesson 11

An Introduction to Integration

Aug 13, 2012

§1 Summary of Differentiation

In the last lecture derivative of function was defined and rules of differentiation were discussed. We briefly recall the definition and the rules here.

Definition 2. *If the limit*

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \tag{11.1}$$

exists, its value is defined to be the differential coefficient of function $f(x)$ and is denoted by $\frac{df}{dx}$. Alternative, shorter, notation $f'(x)$ is also used to denote the derivative of a function $f(x)$.

Rules of Differentiation

Let $f(x), g(x), ...$ be functions which are differentiable at x and $\alpha, \beta, ...$ be constants. Then we have

1. $\dfrac{d}{dx}(\alpha f(x)) = \alpha \dfrac{d(f(x)}{dx} \qquad \alpha \in \mathbb{R}$,
2. $\dfrac{d}{dx}(f(x)+g(x)) = \dfrac{df(x)}{dx} + \dfrac{dg(x)}{dx}$,
3. $\dfrac{d}{dx}(f(x)\cdot g(x)) = g(x)\,\dfrac{df(x)}{dx} + f(x)\cdot \dfrac{d}{dx}g(x)$,
4. $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{f'(x)g(x)-g'(x)f(x)}{(g(x))^2}$ if $g(x)\neq 0$,
5. As a special case of the last result, taking $f(x)=1$, we get $\dfrac{d}{dx}\left(\dfrac{1}{g(x)}\right) = -\dfrac{g'(x)}{(g(x))^2}$, if $g(x)\neq 0$.

6. Chain rule : $\dfrac{df(g(x))}{dx} = \dfrac{df}{dg}\dfrac{dg}{dx}$.

There are two ways of introducing integration.The first route to integration, as limit of a sum, a process to compute area under a curve, leads to definite integrals. The other route is to regard the integration process as inverse of differentiation and to to introduce the (indefinite) integrals as anti-derivative. The relation between the two approaches is provided by the Fundamental Theorems of Calculus.

§2 Definite Integral as Area under a Curve

Need for integration arose from the problem of computing area under a curve. Fig.18 shows a curve $y = f(x)$. Suppose we are required to compute the area between the curve, x- axis, lying between $x = a$ and $x = b$. We divide the interval (a, b) into a large number of parts by means of point $x_0, x_1, x_2, ..., x_k, x_{k+1}, ..x_n$ where $x_0 \equiv a, x_n \equiv b$. The area under the curve with x ranging from x_k to x_{k+1}, shown as shaded area, can be approximated by $f(x_{k+1/2})(x_{k+1} - x_k)$ where $x_{k+1/2}$ is a point between x_k and x_{k+1}. Summing over all the parts the total area is approximated by

$$\sum_{k=0}^{n-1} f(x_{k+1/2})\Delta x_k, \qquad \Delta x_k = (x_{k+1} - x_k) \tag{11.2}$$

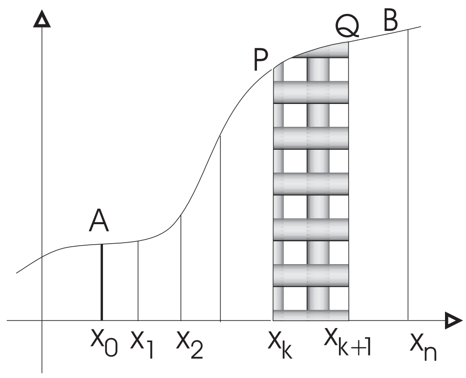

Fig. **18.**

Taking the limit in which the number n of intervals goes to infinity and the length of each interval Δx_k shrinks to zero, we would get the area as

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_{k+1/2})\Delta x_k \tag{11.3}$$

If the above limit we say that the function is integrable in the interval $a \le x \le b$ and the value of the limit is just the definite integral denoted by $\int_a^b f(x)\,dx$.

§3 Integral as anti-derivative

Definition 3. *Let $F(x)$ be a function whose derivative exists at a point x and equals a function $f(x)$, then we say that $F(x)$ is antiderivative of $f(x)$ and we write*

$$F(x) = \int f(x)\,dx + C \tag{11.4}$$

where C is a constant. As a consequence every known formula and result on differentiation can be turns into a corresponding formula or result.

We begin with a list of a few important basic rules of integrals, following from the corresponding results on differentiation.

Integral of multiple of function and sum of two functions

$$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx \tag{11.5}$$

$$\int \alpha f(x)\,dx = \alpha \int f(x)\,dx \tag{11.6}$$

Integration by Parts The derivative of product of two functions $f(x), g(x)$ is given by

$$\frac{d}{dx}(f(x)\cdot g(x)) = g(x)\,\frac{df(x)}{dx} + f(x)\cdot\frac{d}{dx}g(x) \tag{11.7}$$

This equation can be turned into a rule for integration. Let $F(x)$ and $G(x)$ denote the antiderivative of the functions $f(x)$ and $g(x)$ respectively, so that

$$\frac{d}{dx}F(x) = f(x), \qquad \frac{d}{dx}G(x) = g(x) \tag{11.8}$$

$$\int f(x)\,dx = F(x) + C \qquad \int g(x)\,dx = G(x) + C \tag{11.9}$$

The derivative of the product $\frac{d}{dx}\Big(f(x)G(x)\Big)$ gives

$$\frac{d}{dx}\Big(f(x)G(x)\Big) = f(x)\frac{dG(x)}{dx} + G(x)\frac{df(x)}{dx}. \tag{11.10}$$

Taking anti-derivative gives,

$$f(x)G(x) = \int f(x)\frac{dG(x)}{dx}\,dx + \int G(x)\frac{df(x)}{dx}\,dx \tag{11.11}$$

$$\therefore f(x)G(x)\,dx = \int f(x)g(x)dx + \int \frac{df(x)}{dx}\times G(x)dx, \tag{11.12}$$

$$\Rightarrow \int f(x)g(x)\,dx = f(x)G(x)\,dx - \int \frac{df(x)}{dx}\times G(x)\,dx. \tag{11.13}$$

Substituting $\int g(x)\,dx$ for $G(x)$ gives the "rule for integration by parts"

$$\boxed{\int f(x)g(x)\,dx = f(x)\int g(x)\,dx - \int\left(\frac{df(x)}{dx}\times\int g(x)\,dx\right)dx} \tag{11.14}$$

This rule written in words becomes

Integral of product of two functions
= the first function × the integral of the second function
− integral of (derivative of the first function ×
the integral of the second function)

§4 Examples

Example 5:

(5.a). We know that $\frac{d\cos x}{dx} = -\sin x$ hence taking antiderivative we get

$$\int \sin x\,dx = -\cos x + C$$

(5.b). The differential cofficient of $\sin x^2$ w.r.t. x is $2x\cos x^2$, hence

$$\int 2x\cos x^2\,dx = \sin x^2 + C$$

§5 Fundamental Theorem of Calculus

We have talked about the integral as antiderivative and the definite integral. The relation between these two is given by the fundamental theorem of calculus.

Theorem **1** (Fundamental Theorem of Calculus Part-I)**.** *If f is a continuous function on $[a,b]$, then the funcion $F(x)$ defined by*

$$F(x) = \int_a^x f(x)\,dx$$

has a derivative at every point of the interval $[a,b]$ and

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(x)\,dx = f(x) \qquad a \le x \le b \tag{11.15}$$

Theorem **2** (Fundamental Theorem of Calculus Part-II)**.** *If f is continuous at every point of interval $[a,b]$ and F is any anti-derivative of f on $[a,b]$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{11.16}$$

Note that the anti-derivative of f is defined up to an arbitrary additive constant, but this constant cancels out in the right hand side of the above equation and the answer for $\int_a^b f(x)\,dx$ does not contain any arbitrary constant.

Lesson 12

Kinematics of Rectilinear Motion

Aug 22, 2012

§1 Plan of Rest of The Course

From this lecture onwards I will try to follow the book by H.C. Verma (HCV) as closely as possible. It does not necessarily mean that every thing I do will be from the book. What it means is that I will follow that organisation and club things from different chapters together which are similar, or require a common concept or a set of common skills. There will be new material which is not available in HCV. Most of the problem solving will be from this book because there is no need to go to any other book. This book has a fairly good range of problems and solved examples. I suggest that you bring the book at least on Mondays during the tutorials.

Let me now give you the plan of the remaining lectures of the courses.

- KINEMATICS
  Main Reference Chapter 3 of HCV Verma.
- NEWTON'S LAWS AND APPLICATIONS;
  This will cover topics from several chapters, Chapters 4,5,6,7 of HCV.
  I will select problems from these chapters which illustrate use of Newton's laws in different settings. It will include some problems from Chapter 11 on gravitation also.
- MOTION IN NON INERTIAL FRAMES
  Pseudo forces, Accelerating Frames; Lift problems
  Rotating Frames, Centrifugal and Coriolis force and its consequences.
  Effect of Earth's Rotation
  Not fully available in HCV Verma. All the formalism that has been developed on rotation will be used here. The central result on rotation of axes will be needed. Details of the derivation of the formula will not be required.
- WORK AND CONSERVATION OF ENERGY
  Work energy theorem. Conservative and Non-conservative forces;

Potential Energy; Potential energy of springs, gravitational field; Escape Velocity
Conservation of Mechanical Energy;

This discussion will be based primarily Chapters 8. Some amount of gravitation, using conservation of energy, from Chapter 11/HCV will come here. Discussion of simple harmonic motion from Chapter 12/HCV will come here; I will include an introduction to small oscillation about equilibrium; This part is not available in HCV. A discussion of normal modes of coupled oscillations does not require any extra preparation and will therefore be introduced here. A few examples on coupled oscillations and normal modes will be given. You will learn these topics in more detail in the course on waves and oscillations.

- CENTER OF MASS AND CONSERVATION OF LINEAR MOMENTUM;
  Definition of centre of mass. Principle of Momentum Conservation; Elastic Collisions

- CONSERVATION OF ANGULAR MOMENTUM AND ROTATIONAL MECHANICS
  My treatment of this topic will be very different from HCV. I will present important general results and use them to discuss the special case of rotation about a fixed axis. You must have already had some introduction to rotational motion when the axis of rotation is fixed, as is the case in the experiment on fly wheel. Before coming to the rigid body dynamics, I will begin with a discussion of systems with a number of particles.

  The central result in this topic is decomposition of the motion into the motion of center of mass and a rotation about the center of mass. Kinetic energy and angular momentum of many particle system have a corresponding decomposition. In cases, when net torque on a system is zero, we have conservation of total angular momentum. Applications will include a proof of Kepler's three laws of planetary motion. As a simple consequence of conservation of angular momentum, it will shown that the planets move in a plane and that the aerial velocity is constant (Kepler's Second law). That the planets move in elliptic orbits, and Kepler's third law, will be derived using the conservation of energy and angular momentum. Most of it is not available in HCV Verma.

- RIGID BODY DYNAMICS
  It will of course include rotatory motion about a fixed axis. The treatment will be general and will go beyond simple fixed axis case. A discussion of moment of inertia tensor, angular velocity of a rigid body, kinetic energy and angular momentum of rigid body will be presented. We will discuss some illustrative applications of the rigid body motion. This part will again require use of transformation of coordinates under rotations. The topics discussed are not available in most books at the level of HCV and Resnick and Halliday.

As far as mathematics requirement is concerned, it will be developed as and when needed. Time in extra classes utilised for practise for those for whom the lecture presentation may not be sufficient.

I will present a discussion of important concepts, results, and solve examples to develop problem solving skills that are needed. Since you have all gone through a course on mechanics at the level of HCV, I strongly suggest that for topics in future lectures you revise the concepts and summarise important formulae needed in advance before the topic is taken up in the class. So We will now begin with the first item KINEMATICS.

[Q/Me]:= What do we mean by kinematics? What do you understand by kinematics? How is it different from dynamics?

Kinematics is study of motion, relation between displacement, velocity, acceleration etc. An understanding of mathematical definition of these concepts is needed. There will be no reference to forces and Newton's laws. A special case of constant acceleration will be taken up in detail.

Dynamics is study of forces, and the kind of motion that will take place in presence of a given set of forces. In dynamics, the Physics component comes in and the laws of motion are required.

I believe that all of you had some introduction to mechanics at the level of HCV. So I will not repeat everything that is given in HCV or Resnick Halliday or any other book. I will discuss important concepts, summarise important facts and results and solve problems aimed at developing basic skills that you may have to learn. If you have any difficulty ask questions. Merely repeating everything will not be best use of class time and will not be very effective and can become very boring. I do understand that some of you may want certain things in more details, so you should ask questions; it is primarily your responsibility to ask for details.

§2 Distance and Displacement

We begin with a discussion of distance and displacement.
Since you are already familiar with these concepts, let me hear from you.
What is distance travelled and what is displacement? Please say in as many different ways as you can. what is common in them? and what is the difference between the two?

Let me have your responses. I want every one to participate.

The responses of the class along with my comments are listed on the next page.

Your responses on the next page show that most of you are thinking about motion in one dimension. Also I suggest that you must get used to thinking in terms of vectors and using calculus terminology as much as you can.

We adopt $\langle 9\rangle, \langle 11\rangle$, see table below, as the definition of displacement. A detailed discussion of the above statements is given separately. If a particle follows a path $APQRS..B$ from initial point A to a final point B with position vectors $\vec{r}_A, \vec{r}_B$ respectively then the distance is length of the full path $APQRS..B$ and displacement is $\vec{r}_B - \vec{r}_A$.

SNo	Responses of the Class	Comment
⟨1⟩	Distance is equal to the length of the path covered. Displacement is the shortest distance between initial and final points.	The second part of the statement is wrong. Displacement, being vector quantity cannot be equated to, or compared with, a scalar quantity such as distance.
⟨2⟩	Distance is a scalar quantity where as the displacement is a vector quantity.	100% correct (But this does not tell us what they are)
⟨3⟩	Displacement is the shortest distance between initial and final points with direction specified.	Correct, but not stated very precisely and clearly.
⟨4⟩	Distance is always greater than zero; the displacement could be negative or zero.	The second part OK in one dimension only;
⟨5⟩	Displacement $\leq$ Distance.	Displacement, being vector quantity can not compared with, a scalar quantity such as distance.
⟨6⟩	Distance depends on the path where as the displacement depends on the end point only.	Agreed
⟨7⟩	Distance equal to zero $\Rightarrow$ displacement=0; but Displacement=0 $\not\Rightarrow$ distance = 0.	Correct; Not very useful property
⟨8⟩	If P and Q are the initial and the final points of a path of a particle, displacement is a vector whose direction is along PQ and has magnitude is equal to the length of line PQ.	100% Correct and complete statement about displacement
⟨9⟩	Displacement is the vector $\overrightarrow{PQ}$.	Same as ⟨8⟩, but better said than ⟨8⟩.
⟨10⟩	Displacement depends on initial and final points only. So it is a state function	OK; But please explain what do you mean by state of a particle in mechanics
⟨11⟩	Displacement is equal to the vector difference $\vec{r}_f - \vec{r}_i$ of the position vectors of initial and final points.	Same as ⟨8⟩ and⟨9⟩ ; Most clear way to define the displacement.

§3 Speed and Velocity

Let us now discuss two related concepts in mechanics, *viz.* speed and velocity.

Give me your statements about speed and velocity. I would like to you to compare these two concepts. Tell me what are their similarities and how are they different? What are the ways we can explain velocity and speed.

Your responses will be listed along with my comments after receiving all the responses.

SNo	Responses of the Class	Comment
⟨1⟩	Speed is a scalar quantity where as velocity is a vector quantity.	Correct
⟨2⟩	Speed is distance covered in unit time, velocity is displacement per unit time.	Agreed
⟨3⟩	Speed is path dependent. Velocity is a state function.	Path dependent ? In what sense? How about velocity? What is exactly meant by state of, say a point particle?
⟨4⟩	The slope of displacement vs time curve is velocity. Area under velocity vs time graph is displacement.	In one dimension, YES; In higher dimensions?
⟨5⟩	Velocity =Displacement/ total time taken	?? Not Quite,
⟨6⟩	Velocity and speed must always be finite, cannot be infinite. Slope of displacement vs time graph $< 90^0$.	Correct , but not a useful when interpreting the graphs.
⟨7⟩	Velocity $= \lim_{t_2 \to t_1} \left(\frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} \right)$. Speed $= \lim_{t_2 \to t_1} \left(\frac{\lvert \vec{r}_2 - \vec{r}_1 \rvert}{t_2 - t_1} \right)$.	Correct
⟨8⟩	Constant speed implies uniform motion; constant velocity does not imply uniform motion.	Both parts wrong!
⟨9⟩	If |displacement| = distance, then |velocity| = speed.	Hmmm..., Thinking
⟨10⟩	|velocity| = speed.	Correct relation bewteen the two.
⟨11⟩	Velocity $= \frac{d}{dt}$ displacement and speed $= \frac{d}{dt}$ distance.	100% correct; Agreed
⟨12⟩	Both velocity and speed have the same units	Agreed

More Comments

- I find that many of you are not participating in the discussions. This is unacceptable. I will have to find ways of ensuring participation by every one. After all these concepts are not new to you, you have been hearing about them from your LKG classes.
- As before, my first comment is that many of you are thinking "in one dimension"; many of your responses apply to motion in one dimension. Let us accept $\langle 7 \rangle$, and use it as the definition of speed and velocity.
- I repeat that you must make best use of vectors and calculus in your thinking.
- It is interesting to note that in response $\langle 3 \rangle$, an attempt has been made to use the concepts of "path independence" and "state". Most likely you have come across the state concept in thermodynamics. What is meant by state in the present context? This issue will be taken up in one of the later lectures, where the structure of mechanics will be briefly discussed.

  Sakhi liked response $\langle 3 \rangle$ and has insisted that I should tell this to my students.
- One of you asked if both expressions

  $$\lim_{\Delta t \to 0} \left| \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} \right| \text{ and } \lim_{\Delta t \to 0} \left(\frac{|\vec{r}(t+\Delta t)| - |\vec{r}(t)|}{\Delta t} \right)$$

  define speed correctly? I need more time to think and defer my answer at this stage.
- Sakhi asks why do we hear about speed of light most of the time? Why do we not talk of velocity of light in the same way as we do it for particles? It is an interesting question and I leave it you to think about it.

Lesson 13

Structure of Mechanics-I States, Laws of Motion

Aug 29, 2012

§1 About Monday Classes and Problem Sessions

From now onwards we will have the following arrangement for the classes. On Wednesdays and Fridays there will be lectures and on Mondays both the lectures will be reserved for problem solving sessions. Afternoon class we will have tutorials like we had on the last Monday. This means that the problems will be given to you in advance and you have to solve the problems in the class and submit before leaving. In addition one person will be called to solve one question the problem on the board, then next problem will be solved by another person an so on. Others will have the choice of doing it yourself on the paper or of following the solution being worked out on the board. The person who comes to the board will get all the help from me and from Vijay. This will ensure that every one finishes the problems in time. I will call you randomly and will make sure that every one comes to the board at least once and solves at least one problem eventually sometime during the semester. So that is the arrangement for the afternoon session. What you do in the afternoon session is to be submitted. All tutorials will be evaluated and returned. They will make up one unit of assessment of 20 marks.

Morning sessions will be something similar again. We will be solving problems except that you work in groups. Again the problems will be given in advance. What is discussed in the mornings need not be submitted. Last Monday we could discuss only one problem in one hour that is not very efficient. I want this discussion session to be kept under control. So what I suggest is the following. For the morning sessions you work on the problems in groups, discuss and thrash them out in advance. One member of a group will be asked to present solutions of one or two problems, then next a member from some other group chosen randomly will be asked to solve next problem and so on. This is primarily to encourage group activity. You should have some amount of activity discussing Physics among yourself. In India we lack team work. In many organizations people are expected to work as a team. You can see results where ever there is team work. You are expected to solve the problems in advance and come prepared to present solutions. This will also help you to prepare in facing an audience of fifty to sixty and in making a presentation of your ideas to them. The solutions will have to be presented and explained to the class as if you are the teachers and you are explaining it to the class. This part will be evaluated in some way which will be known to you on Monday. The part that should make it interesting for you is that the whole class will be participating in evaluation in the morning sessions. Thus all of you will have to submit a evaluation report for every Monday morning session. This will be considered as group activity and your evaluation will be included for grading. All group activities put together will constitute one unit of evaluation and will carry 20 marks.

The problems in the afternoon sessions will be primarily from H.C. Verma's book. The problems in the morning sessions will be on the lectures which I will be giving on Wednesdays and Fridays. The problems in the morning sessions will be similar to what you have solved in the tutorial on rotations and you may not find them in any other book. These problems will be designed to give you supplementary material and provide you help in understanding the lectures.

The other two lectures, on Wednesdays and Fridays will constitute a complete, coherent course by itself, delivered in a logical order. It will be at a slightly higher level. An attempt will be made to consolidate and upgrade what you have already done in 12th standard. Also I will make an effort to present new topics which are not really difficult and are accessible to you with your level of Mathematics and Physics. Many of these problems do not form part of the text books at 12+ level. Traditionally, the topics, I hope to cover, are included in higher level courses, typically in M.Sc. or B.Sc. honours level courses in Mechanics. What ever can be brought to your level will be included. You do not get worried in the sense that the course will become Hi-Fi or an advanced course. I am only trying to optimise the returns you get for what you already know.

I am attempting to take you to higher level without being too demanding on preparation needed. The results on rotations that have been derived for you is an example, it makes use of only vector algebra and some calculus and does not use any matrices. Normally, I have presented this part of lectures using matrices to M.Sc. students. Only yesterday I discovered that the central result on transformation of coordinates under a rotation can be made much simpler and transparent and now it will be part of an assignment.

I will also make an effort to put a draft of lecture notes in advance on the Moodle Course Site. Later the draft will be modified and will be brought as close to the class room lecture as is possible. I will try to synchronise the lecture delivery in the class and the internet and reduce the phase difference as much as possible. Right now I have already put some lecture notes. Some of them are older versions. Some others are up to date. What I put up on the internet will be almost same, within about 95 to 98% of what I do in the class. So that you get everything. There may be some improvements and and some 2 to 3 % additional remarks may be added. The only problem is time lag and I will make every effort to minimize it.

Let us then begin with the main topic of discussion

In the Newtonian formalism, the Newton's Law play a central role. We have seen that in order to use the laws to predict the behaviour of mechanical systems, we need to introduce frames of reference. It was also noted that the use of vectors offers important advantages.

§2 Structure of Physical Theories

Here we give an overview of the general structure of formalism of Newtonian Mechanics. There are following four major components to be described in the language of the Newtonian Mechanics.

1. State of a physical system
2. Physical quantities of the dynamical variables
3. The laws of motion or the laws of evolution
4. The forces or the interactions

States of Physical system

The idea of *state* of a physical system has already been introduced in the previous lecture. One of you had observed that the displacement is a state function and does not depend on the path.

What is meant by state of a system? If you are already familiar with the concept I will build upon your understanding.

Dialogue with Students

1

Q/Me := Where did you first come across the state in Physics for the first state?

A/S := *Thermodynamics.*

R/Me := Yes, in thermodynamics, you must have come across the concept of state function. Entropy is a state function and internal energy is a state function. What does state mean to you?

A/S := *State function is something that depends on the initial and final state.*

R/Me := That is Correct. But my question is what do you understand by state?

R/S := *Whatever we can measure is the state. For example, we can measure temperature and that is the state.*

R/Me :=*Let us be clear. Do you mean to say that temperature is the state?* Entropy is description of the state? Is that what you want to say?

A/S :=What ever parameters I can measure at an instant of time that is the state of the body.

Q/Me :=So shall I say that these parameters are the state of the system? Is it correct to say that the **temperature is** the state of a thermodynamic system? entropy is another state?

A/S :=*Whatever parameters I can measure at an instant of time is the state. The parameters are not the state but they give us the state.*

R/Me= That is almost correct. So shall I say that if I take a "snapshot" of the system at a time that gives the state of the system. Any other comment?

A/S := *Properties of the body.*

R/Me := State is not a property of the system, it is concept. I am not asking a question, I want to know what is your understanding of the state?.

R/S :=*At a particular time whatever properties of the of the body we can measure is the state.*

R/Me :=You have some idea. Let us try to make it more precise. OK Let us continue with our discussions.

A/S :=*Surroundings.*

A/S :=*No, environment is not part of the system in thermodynamics. So state concept cannot be defined in terms of something that is not part of the system.*

A/S :=*Position of the body at particular time?*

R/Me :=Position is not the state of the body? In thermodynamics there is no position, so? Position does not make sense for all systems, the concept of state does. So?

A/S :=*State could be defined as set of all parameters that can be measured at a time.*

The state concept is an important concept that will keep coming again and again. It is the information about the system that is needed to specify it at a particular time. The specification of the state of a point particle in Newtonian mechanics at any time, t_0, means specifying the values of its position vector, $\vec{x}$ and momentum vector $\vec{p}$ at that time. One criterion is that knowing the state at any time enables us to compute every other observable quantity such as energy momentum angular momentum of the system.

Let me take an example that is familiar to you. The example of an ideal gas. For this case the state is completely specified by giving any two of the three quantities pressure, volume and temperature. If you know temperature and volume you can compute pressure or internal energy or any other thermodynamic function. That forms one requirement on specification of the state.

The other criterion is that state specification should be such that knowing the physical laws, given the laws of dynamics, it should be possible in principle to find the state of the system at a later time.

These two criteria which specify the state of a system. So I summaries the two requirements:

1. The state is complete specification of the system at a time, complete in the sense that any other physically measurable quantity of the system can be computed.

2. Using the equations governing the time evolution and the knowledge of the system at a particular initial instance of time allows us find the state at a later time .

Let me now give consider examples. Consider the simplest example of a point particle.

Q/Me :=What will specify the state of a point particle at a particular ?

A/S :=*Position.*

Q/Me :=If you give me position at a given time? Is it a complete specification of the point particle? No, what else do we need?

A/S :=*Velocity.*

R/Me :=That's right. I should also specify the velocity to know the state completely. Anything, I will be able to compute such as kinetic energy, angular momentum..

Q/S :=*Acceleration?*

R/Me := Acceleration should not be included. The acceleration is determined by the forces and cannot be specified. It is not needed either. Note that the equations of motion contains second order derivative of position w.r.t.time. Hence the position and only first order derivative are needed to find the state at a later time. Acceleration is not part of state specification.

For a simple pendulum the state is completely specified by the angle that the string makes with vertical and the angular velocity at that position. It can also be specified by means of the horizontal displacement and the velocity.

As other examples of specifying the state, for a system of n- particles, the state will be *completely specified* by giving position and momentum vectors of all the particles.

Q/Me :=Now we will take a little more complicated example. May be you can think of some examples.

A/S :=*A moving cart*

A/S := *A satellite*

Q/Me :=In all these examples we treat the bodies as point particles unless we want more details.

A/S :=*Rotating rigid bodies.*

Q/Me :=This is a good example. Examples of rigid bodies are flywheel, spinning top. We will discuss it later in the course.

Q/Me :=Charged particle, that is also a point particle. This is slightly different kind of system. I will keep it. From the point of view of mechanics, it is not different from a point particles.

R/Me :=Give me more complicated example.

OK Let us proceed. For a rigid body, we will see that besides the position and momentum of the centre of mass of the body, one needs to consider three parameters that specify the orientation of the body and its angular velocity to specify the state of the system.

A more complex example is that of a vibrating string held fixed at the end points.

Q/Me :=How do you specify the state of a vibrating string.

A/S :=*Amplitude*

A/S :=*Tension*

A/s :=*Frequency*

A/S :=*Displacement*

A/S :=*Wave equation*

A/S :=*Time period*

R/Me :=Amplitude at what point? Tension will determine the forces and is not part of state specification. Wave equation is the physical law derived governing the vibration of strings. Time period is a property of a particular mode of vibration. Also several, more than one, time periods may be associated with a single mode of vibration.

A/S :=*Amplitude at all positions.*

R/Me :=Amplitude? I suppose you mean displacement!. This alone is not sufficient. If we know the position of a point particle at one time can we predict its motion? So?

A/S :=*Velocity is needed.*

R/Me :=Yes! Similarly, for a vibrating string we need the velocity, along with its position, of each point of the string to specify its state.

Thus we need to know the displacement and the velocity of each point of the string. How many pieces of information are needed for a vibrating string? One, two, three, four ? How many ? We need infinite number of values, two for each point of the string. We need position and velocity of each point and the number of points of the string is infinity.

The state of different systems will be specified in different ways. For the same system it may happen that the state can be specified in several different ways, remember the ideal gas.

Different theories may be needed to explain different phenomena involving the same system. In such a case, the answer depends on the system but also what theory is needed to describe the system. For example, we are studying mechanics now. Today, for an electron going round the nucleus the state of the electron is given by its position and velocity. But in the fourth semester, when you learn quantum mechanics, you will describe the state of the electron by a "wave function".

Physical Observables

These are functions of coordinates and velocities and are measurable quantities such as energy momentum and angular momentum.

Laws of Motion and Time evolution

The Newton's laws of motion govern the behaviour of mechanical systems. These laws tell us when the system will be in equilibrium and when it will be moving or changing with time. The specification of the state refers to the system at one time only. Most often we are interested in the changes that take place with time, how position or velocity is changing with time. The laws of a physical theory, such as Newtonian Mechanics, tells all about this change, also referred to as *time evolution. Predicting time evolution of a system means finding out the state of the system at a later time t, when the state of the system is known at an initial time t_0. The three Laws gives us all the information covering the 'rules of the game' to obtain the time evolution of a system.*

Interactions

For a physical system the knowledge of the laws of motion and the state of the system at initial time alone is not sufficient to predict future behaviour of the system. What else is needed. It I give you the position and velocity of the earth at an initial time t_0 and all the mathematical machinery that you want, can you predict the position of the Earth after time t_0 using the Laws of motion? Is there any other information that is needed? The answer is that we need to know about the *force* that the Sun exerts on the Earth to compute the Earth's orbit. The *interactions* is a general term for the concept, such as force, which controls the behaviour of a system and will be different for different systems, and from one situation to another. A charged particle falling in Earth's gravitational field experiences a different kind of interactions than the same particle moving in an electric or a magnetic field.

⊡ It must be emphasised that all the components, described above, will be present in all the major formalisms of physical theories. For example, we need a description of state, laws and EOM, and knowledge of interactions to complete description of systems in all major theories such as the ones listed below.

- Classical Mechanics
  States : Generalised coordinates
  Laws, EOM : Action Principle Lagrangian EOM, Hamiltonian EOM,...
  Interactions : Potential Energy, Lagrangian, Hamiltonian or Action

- Statistical Mechanics, Thermodynamics
  States: Variables such as pressure, temperature and Volume for a gas
  Laws, EOM : Laws of Thermodynamics, ... Interactions : Hamiltonian

- Quantum mechanical system,
  States: Wave function ;
  EOM: Schrodinger equation;
  Interactions: Hamiltonian, Lagrangian

- Charges in presence of electromagnetic fields

▷ A few questions for you to think :

1. What will be needed to describe the state of torsional pendulum at a given time? state of vibrating string at a given time?
2. Do you think equations of motion for a vibrating string can be derived from Newton's Laws?
3. You have an experiment on torsional pendulum in Physics-1 lab and an experiment on Kater's Pendulum. Do you think that the EOM of motion for these also can be obtained from the Newton's Laws?

Lesson 14

Structure of Mechanics-II Forces, Inertial Frames

Aug 31, 2012

§1 Introduction

First thing first: I want all those students, who did not have mathematics in the 11th and 12th classes, to rate their own level of confidence in different items on topics of mathematics on a scale of 1 to 5 on the sheet given to you. Please return the sheet at the end of the lecture. And any one who does not feel confident and wants to join extra classes for these topics should also give feedback. This will give me a good idea what should I cover and how much time should be devoted to different topics.

We were discussing the structure of classical mechanics. All physical theory, such as thermodynamics, and electro-magnetic theory, and quantum mechanics, and of course mechanics included, have the following important components in common:

- Description of Physical states;
- Dynamical Variables or observable physical quantities;
- Physical Laws, or Laws of Motion, or time evolution; This part tells us how a system changes with time?
- Interactions, Forces.

The first three components are general in nature and applicable to a large class of systems within the domain of application of a theory. While the laws such as Newton's laws are stated in general terms and apply to all mechanical systems, the specific form of interactions, such as forces in mechanics are different from system to system and is needed to study behaviour of a system of interest.

We will be interested in three types of physical systems of interest.

1. Systems with finite number of particles.
2. Rigid bodies which can be thought of as systems consisting an infinite number of particles, such that the distances between the particles are held fixed.
3. Systems, other than rigid bodies, such as strings which require an infinite number of variables to specify the states.

In this course we will be interested in systems of many particles and dynamics of rigid bodies.

The vibration of strings is will be discussed in detail in the course on wWaves and Oscillations. The time evolution of vibrating strings is governed by the wave equation which is derived using the Newton's laws and the facts known in elasticity.

Let me repeat that the description of state at a given time t means complete information about the system, complete in the following two respects.

- At the time the state is specified, I should be able to compute any thing I want to know about the system at that time. So far example knowing the position and velocity of a point particle, every other quantity of interest such as momentum, kinetic energy, potential energy or angular momentum can be calculated. So for a single particle specifying its state means specifying the the position and the velocity. Notice that the displacement is of not interest any more and has been replaced by the position vector.
- The state specification should also be complete in the sense that information about the system at an initial time should be sufficient for knowing the state at any later time. If the system is in equilibrium, there is nothing more to be done. If the system is changing with time, then this information about the state at a time, when used along with the knowledge of the interactions *i.e.* the forces and the laws governing the time evolution, *i.e.* the equations of motion, should be sufficient to derive the information about the state of the system at a later time.

The concept of dynamical variables should be clear to you by now. The concept of path dependence of physical concepts will appear when we discuss potential energy. For the time being we will not talk about it.

§2 Frames of Reference

Next let us come to the laws of motion. For the purpose of this course the laws of motion will mean the three laws given by Newton. It could be some other physical principle equivalent to the three laws of motion. You should not think that Newton's laws are the end of story for mechanics. The same laws have been reformulated in several different ways. These reformulations make the structure of mechanics richer and more useful and have made it possible to go beyond Newtonian mechanics.

The three laws of motion given by Newton are valid in special class of frames of reference called *inertial frames*. What do I mean by a frame of reference?.

What is your understanding of the concept of a frame of reference. What is the kind of idea you have about the frame of reference? You may not be able to say it very precisely, but you must have some feel for it. So let us discuss frames of references. Tell me what do understand by specifying a frame of reference?

The concept of frame of reference will keep coming again and again. Specially when you learn special theory of relativity. The laws of motion will change, certain other concepts will change drastically but the concept of inertial frames will continue to occupy an important place.

Q/T:= So what is a frame of reference? and what is an inertial frame?

A/S:= *It consists of a coordinate system which is used to observe a system.*

R/T:= YES, it is almost the correct answer.

We have a theory, the Newtonian mechanics. What do we want to do with the theory? Initially, the physicists observed natural phenomena and and from a study of these phenomena, physical laws have been postulated. Next we want to make predictions about behaviour of systems. We want to test correctness of the postulates against experiments. Usually an enormous amount of effort over long periods is required to identify the central concepts, to check if the postulates pass rigorous tests against experiments. Only after the test are passed, we arrive at a theory that is accepted and is considered as established within a well defined domain of applications.

Once a theory has been established, it can be used to predict behaviour of the system. One can also make observations on the system and then make a comparison the theory and experiments. Ultimately, we want to move on to design systems for specific purposes and applications. Some examples are bicycles, buses, cars, aeroplanes, cranes etc, which will behave in particular way to meet specific requirements.

For all these tasks, we need to translate our equations into mathematical form which will allow us to compute numbers. In mechanics we need to describe every particle in terms of its coordinates and velocities, or any other quantities of interest, and these are computed from the positions and velocities. The final answers theoretical as well as experimental, have be expressed in terms of numbers. We may want to make a detailed comparison of theory with experiments, or to design a satellite to be sent on a mission to Mars, we need to introduce the coordinate system, a system of axes; the use of position operator entirely in geometrical language, welcome for many theoretical purposes, is inadequate for this purpose.

Thus a frame of reference is a set of coordinate axes w.r.t. which positions and velocities of all particles are measured. *We also add a clock, a device to measure time intervals, to the frame of reference keeping the special theory of relativity in mind.* In the Newtonian mechanicas all clocks are assumed to run at the same rate. Later when you learn the special theory of relativity in the fourth semester, you will know that the clocks run at different rates for different observers. Then a clock becomes an essential part of a reference frame.

So far this course has been in the third year and was not compulsory for every one. For your batch special relativity, also quantum mechanics, comes as a part of fourth semester Physics course. So every one, irrespective of your chosen stream, will have a chance to get an introduction to special relativity and also quantum mechanics.

Thus a frame of reference, then, consists of a clock to be used to measure time intervals and a set of axes which is used as a reference for measuring distances and specifying coordinates. The clock is unimportant as the time intervals are same in all frames, and the rate at which the a clock runs does not depend on whether the frame is inertial or non inertial. The time is same for all the observers.

§3 Inertial Frames

Out of all possible frames, the inertial frames occupy an important place in mechanics. How do we define an inertial frame. One can take the first law as the definition of inertial frame. An inertial frame is one in which Newton's first law is valid. This is somewhat circular definition. It is defined as a frame in which all bodies have uniform motion in absence of forces. How do we check that a frame is inertial or not. We have to use known forces and predict the behaviour of a mechanical system. If the observed behaviour agrees with that predicted by Newton's laws, we conclude that the frame of reference is an inertial frame, otherwise non inertial.

Let us agree that there exists at least one inertial frame. Any other frame moving with uniform velocity w.r.t. the first frame is also an inertial frame. The Newtonian laws are the same in all inertial frames. The velocity of an an inertial frame cannot be measured by means of mechanical experiments entirely conducted on motion of objects w.r.t. the frame.

On the other hand, the acceleration of a frame can in principle be measured by means of suitably designed experiments; so acceleration of a moving train or a car be measured by doing an experiment conducted entirely within the train or a car. I can simply not sit back on my chair and say that my frame is an inertial frame of reference; whether a particular frame is inertial or not is ultimately decided by experiment. All such experiments will always have some limitation. The results will have an accuracy up-to which the conclusions will apply. For a frame acceleration may be negligible. Whether we may regard the frame to be inertial or not, will finally be decided by the context in which we want to do the calculations and what kind of applications of mechanics are being considered. So for example, motion of auto mobiles and other vehicles on the earth and for purpose of our experiments in laboratories, the effect of non inertial nature of frame fixed in the earth can be ignored and the earth can be regarded as inertial frame. For the purpose of launching a satellite one cannot regard the earth as an inertial frame. As another example, it is known that for designing a GPS system one has to go beyond Newton's laws and tiny corrections coming from general relativity to the Newtonian laws must be taken into account.

How good is the earth as an inertial frame? Sitting on the earth we can measure its acceleration w.r.t. a frame of reference spinning with the earth. This can be done by means of a measurements involving a Foucault pendulum. This acceleration is approximately given by $v^2/r \approx 3.4 \times 10^{-3}g$, where g is the acceleration due to gravity on the surface of the earth. This number is obtained by computing $\omega^2 R$ where ω is earth's angular velocity about is own axis and R is the mean radius of the earth.

The frame fixed in the earth and rotating it is an approximate inertial frame. But this is not an exact inertial frame as the earth goes round the sun in an elliptic orbit. The acceleration of the earth due to its orbital motion is $\approx 6 \times 10^{-4}g$. So a frame fixed in the sun and at rest w.r.t. the distant stars is a better inertial frame of reference. The sun itself moves around the centre of our galaxy and has an acceleration $\approx 2.4 \times 10^{-11}g$. So this frame fixed in sun an at rest w.r.t. the distant stars is an inertial frame to a great accuracy.

It is suggested that you may search on internet and get the data needed to compute relevant accelerations as given above and find the accelerations of different frames discussed above. That will be part of an assignment for you.

Suppose I want to do computations in a non inertial frame. Newton's laws cannot be applied in a non inertial frame. *So then how do we do any calculation? We need to first write the equations in an inertial frame and transform the equations of motion to the non inertial frames of interest.* We will discuss this in detail in later lectures.

Lesson 15

Summary of Structure of Physical Theories

Sept 5, 2012

§1 Summary of Structure of Physical Theories

§2 Degrees of Freedom

Examples Discussed

1. System of N particles
2. Vibrating Strings
3. Rigid Body Dynamics

§3 Non inertial Frames of Interest

- Frames moving with a constant acceleration, elevator
- Frames rotating with constant angular velocity
- Frames rotating with angular acceleration; Axis of rotation and the magnitude of the angular velocity may change.

For More Details See Your Class Notes

Lesson 16

EOM in Rotating Frames

September 12, 2012

§1 Non Inertial Frames

I continue our discussion of non inertial frames. Basically there are two types of non inertial frames that are of interest.

- Frames accelerating with a constant acceleration;
- Rotating frames.

The Newton's Laws are not valid in non inertial frames. I cannot write

$$\text{Force} = \text{Mass} \times \text{Acceleration}$$

in a non inertial frame. So the question is how do write the EOM in accelerating and rotating frames? The E OM will look different and what are new consequences of Newton's Laws being different in non inertial frames?

The first case is simple. The examples are that of an lift, car or a train accelerating with a constant acceleration. So how do get the EOM.

I will always K for an inertial frame. In most cases it will clear which frame is inertial. If it is not, we must identify an inertial frame. I assume that there is an observer O who is performing an experiment and taking data. Then we have another frame K' accelerating with a constant acceleration. Let there be an observer O' in the frame K'.

Let $\vec{x}(t)$ and $\vec{x}'(t)$ be the position vectors of a point at time t as seen by the two observers O and O' respectively. Let the positions of the axes in the two frames coincide at time $t = 0$. Also let $\vec{x}_O(t)$ be the position vector of the origin of K' at time t as seen by the observer O.

Place ⟨**Figure**⟩ Here

Then it is obvious that

$$\vec{x}(t) = \vec{x}_0(t) + \vec{x}'(t) \tag{16.1}$$

Differentiating the above equation two times we get

$$\frac{d\vec{x}(t)}{dt} = \frac{d\vec{x}_0(t)}{dt} + \frac{d\vec{x}'(t)}{dt} \tag{16.2}$$

$$\frac{d^2\vec{x}(t)}{dt^2} = \frac{d^2\vec{x}_0(t)}{dt^2} + \frac{d^2\vec{x}'(t)}{dt^2} \tag{16.3}$$

$$\tag{16.4}$$

Therefore the accelerations of a body, $\vec{a}$ and $\vec{a}'(t)$, in the two frames are related by

$$\vec{a} = +\vec{f} + \vec{a}'(t) \tag{16.5}$$

where $\vec{f}$ denotes the acceleration of the frame K' w.r.t. the frame K. Multiplying the above equation by the mass of the body

$$M\vec{a} = M\vec{f} + M\vec{a}'(t) \tag{16.6}$$

$$\text{or} \qquad M\vec{a}' = M\vec{a}(t) - M\vec{f} \tag{16.7}$$

$$\tag{16.8}$$

Using Newton's Laws in the first frame K and replacing $M\vec{a}(t)$ with force $\vec{F}$ we get

$$\text{or} \qquad M\vec{a}' = \vec{F} - M\vec{f} \tag{16.9}$$

The second term $-M\vec{f}$ is called pseudo force. The EOM in the non inertial frame look similar but one has to include the pseudo force to the forces acting on the system. The pseudo forces to be included will depend on how the frame is accelerating. The pseudo force is equal to minus of mass times acceleration of the frame with respect to an inertial frame.

You may ask how are pseudo forces different from real forces. Can one distinguish between real and pseudo forces experimentally? Before answering this question I have to discuss fundamental forces.

Fundamental Interactions

There are four types of fundamental forces or interactions. These are

1. Gravitational Forces
2. Electromagnetic Forces
3. Strong forces which are responsible for binding of a nucleus.

4. Weak forces or the forces responsible for beta decay.

Of these four forces, we experience only the first two type of forces in our daily life. These have characteristic strength, range and other properties. The strong and weak forces have very short range and don't play any role in mechanics. All the mechanical forces are electromagnetic in nature. So in our discussion in this course we will be concerned about these two types of interactions. The electromagnetic forces do not depend on mass, hence the pseudo forces can be differentiated from the electromagnetic forces.

It turns out that it is impossible to distinguish between gravitational forces and pseudo forces. This is because the both gravitational forces and pseudo forces in an inertial frame are proportional to the mass of the body. In practical terms it means that effect of gravitational forces in a given inertial frame can be reproduced by that coming from pseudo forces in an non inertial frame. So for example, EOM in uniform gravitational field is same as motion in a lift accelerating upwards without any gravitational forces. Essentially a consequence of Einstein's equivalence principle, this statement holds for small regions of space.

It should be noted that the pseudo forces on a body are always proportional to the mass of a body.

What will be the EOM in a rotating frame? To arrive at EOM for a body as seen by an observer in a rotating frame, we must start with the relation between position vectors of a point in the two frames.

developed for rotating frames will be needed to understand the results of above three experiments.

§2 Velocity and Acceleration in Rotating Frames

SEC2

In this section I will briefly outline the derivation of EOM in a rotating frame. I will skip all the computational details.

We are interested in obtaining relations between velocity and acceleration of a particle as seen by an observer in a frame K and by another observer in another frame K'. To set up our notation, we assume that the frame K' is rotating w.r.t. the frame K with a constant angular velocity ω about an axis given by the unit vector $\hat{n}$. It is further assumed that the axes in the two sets of frames of reference coincide at time $t = 0$. The angle of rotation at time t is given by $\theta = \omega t$. The quantities $\vec{x}, \vec{v}, \vec{a}$ will denote the position, velocity and acceleration of the particle as *measured by the first observer*. The quantities $\vec{x}', \vec{v}', \vec{a}'$ will denote the corresponding quantities as *measured by the second observer*. Then the components of the position vectors $\vec{x}, \vec{x}'$ are then related by

$$\vec{x}' = \vec{x} - \sin\theta(\hat{n} \times \vec{x}) + (1 - \cos\theta)\hat{n} \times (\hat{n} \times \vec{x}) \tag{16.10}$$

This equation should be understood as a compact notation for three relations obtained by equating components of both the sides. Let the components of $\vec{x}, \vec{x}'$ as seen by the two observers in their respective frames be given by

$$\vec{x} = (x_1(t), x_2(t), x_3(t)) \qquad \vec{x}' = (x_1'^{(t)}, x_2{}'(t), x_3'(t)) \tag{16.11}$$

Then the components of velocities are obtained by taking derivatives w.r.t. t (recall that we have already assumed $t' = t$).

$$\vec{v}\,' = \frac{d\vec{x}}{dt} = \left(\frac{dx_1(t)}{dt}, \frac{dx_2(t)}{dt}, \frac{dx_3(t)}{dt}\right) \tag{16.12}$$

Substituting $\alpha = \omega t$ in Eq.(16.10)

$$\vec{x}\,' = \vec{x} - \sin(\omega t)(\hat{n} \times \vec{x}) + (1 - \cos(\omega t))\hat{n} \times (\hat{n} \times \vec{x}) \tag{16.13}$$

and differentiating Eq.(16.13) w.r.t. time and writing $\vec{v} = \frac{d\vec{x}}{dt}$ and $\vec{v}\,' = \frac{d\vec{x}\,'}{dt}$ we get

$$\begin{aligned} \vec{v}\,' &= \vec{v} - \omega\cos\omega t(\hat{n} \times \vec{x}) - \sin(\omega t)(\hat{n} \times \vec{v}) + \omega\sin(\omega t)\hat{n} \times (\hat{n} \times \vec{x}) \\ &\qquad +(1 - \cos(\omega t))\hat{n} \times (\hat{n} \times \vec{v}) & (16.14) \\ &= \vec{v} - \sin(\omega t)(\hat{n} \times \vec{v}) + (1 - \cos(\omega t))\hat{n} \times (\hat{n} \times \vec{v}) \\ &\qquad +\omega\cos\omega t(\hat{n} \times \vec{x}) + \omega\sin(\omega t)\hat{n} \times (\hat{n} \times \vec{x}) & (16.15) \end{aligned}$$

where we have rearranged Eq.(16.14), by shifting the two terms proportional to ω to the end of the right hand side. Now note that the first three terms in Eq.(16.15) just give the components of $\vec{v}$, the velocity of the particle as measured by the first observer, w.r.t. *the axes of the rotating frame*. We denote these three components by $\left(\vec{v}\right)_r$:

$$(\vec{v})_r \stackrel{\text{def}}{=} \vec{v} - \sin(\omega t)(\hat{n} \times \vec{v}) + (1 - \cos(\omega t))\hat{n} \times (\hat{n} \times \vec{v}) \tag{16.16}$$

and rewrite Eq.(16.15) as

$$\vec{v}\,' = (\vec{v})_r + \sin(\omega t)\omega\hat{n} \times (\hat{n} \times \vec{x}) + (1 - \cos(\omega t))\hat{n} \times (\omega\hat{n} \times \vec{v}) \tag{16.17}$$

In vector $\vec{x}$ appearing in the right hand side of Eq.(16.17) refer to the position vector components as seen by the first observer *his frame of reference*. We express these components in terms of the components $\vec{x}'$ with respect to the second system using

$$\begin{aligned} \vec{x} &= \vec{x}\,' + \sin\alpha(\hat{\omega} \times \vec{x}\,') + (1 - \cos\alpha)\hat{\omega} \times (\hat{\omega} \times \vec{x}\,') & (16.18) \\ &= \vec{x}\,' + \sin(\omega t)(\hat{\omega} \times \vec{x}\,') + (1 - \cos(\omega t)\hat{\omega} \times (\hat{\omega} \times \vec{x}\,') & (16.19) \end{aligned}$$

where $\vec{\omega} = \omega\hat{n}$. This equation has been written down after the exchange $\vec{x} \leftrightarrow \vec{x}\,'$ in Eq.(16.10) and a change in the sign of the angle of rotation. This process correctly gives $\vec{x}$ in terms of $\vec{x}\,'$ and the result Eq.(16.18) can be also derived from Eq.(16.10) by use of vector identities.

Substituting for $\vec{x}$ from Eq.(16.19) in Eq.(16.17) we get after some simplification

$$\boxed{\vec{v}\,' = (\vec{v})_r + \vec{\omega} \times \vec{x}\,'} \tag{16.20}$$

measured by the second observer with components taken w.r.t. second frame of reference. The left hand refers to velocity measured by the first observer but with the components w.r.t. the second frame. Differentiating Eq.(16.15) w.r.t. time once again and repeating the above steps gives relation between the accelerations.

$$\boxed{\vec{a}\,' = (\vec{a})_r + 2\vec{\omega} \times \vec{v}\,' + \vec{\omega} \times (\vec{\omega} \times \vec{x}\,')} \qquad (16.21)$$

I would like to remind you that $\vec{\omega} = \omega\hat{n}$ is the angular velocity of K' as seen by the observer in the frame K. Note the second observer is in the rotating frame of reference and therefore he sees the frame K' as stationary and K rotating with angular velocity $\vec{\omega}\,' = -\vec{\omega}$.

Questions For You

- Use Eq.(16.10) and vector identities to solve for $\vec{x}$ in terms of $\vec{x}\,'$ and prove Eq.(16.19)
- Verify Eq.(16.20).
- Prove Eq.(16.21). It requires a little long algebra.
- If $(\omega_1, \omega_2, \omega_3)$ are the components of the angular velocity vector $\vec{\omega}$ w.r.t. the frame K, what will its components w.r.t. the frame K'?

Lesson 17

Acceleration in Rotating Frames

Sept 14, 2012

§1 Introduction

In my previous lecture a discussion of how to set up equations of motion, in non inertial frames has been started.

The important physical applications of our discussions will be to the following systems.

1. Frames moving with constant acceleration. I will be solving problems in the tutorial sessions.

2. I will also need to talk about a body rotating with constant angular velocity ω. I am sure that you have all solved problems on this in your earlier classes. The basic result here is that the acceleration of a particle moving in a circle with constant angular velocity is $\omega^R = v^2/R$ and is directed towards centre.

   The cases when a body moves in a circle but with varying angular velocity are also of interest.In this case the acceleration has both tangential and normal components.

3. For motion of a body, moving under action of external forces, I will be interested in writing equations of motion as seen by an observer in a frame rotating with a constant angular velocity. Important examples will be motion of bodies on the earth taking into account of rotation of the earth about its own axis.

4. Another important application of equations of motion in the non inertial frames is to the dynamics of rigid bodies, some examples are spinning flywheel, a top and the motion of the earth around the sun.

You have already had some intorduction to writing the problem of writing EOM in a frame moving with constant acceleration. This will now be taken up in problem sessions only.

For other problems listed above we need to use results on trasformation of position vector under rotations. This result alreday derived is given by

$$\vec{x}' = \vec{x} - \sin\alpha(\hat{n}\times\vec{x}) + (1-\cos\alpha)\hat{n}\times(\hat{n}\times\vec{x}) \tag{17.1}$$

This formula may look very intmidating to you, I will present my results in a simple fashion in a special cases and then generalise.

§2 Acceleration of a particle on a circle

I am sure that all of you have come across rotating frame and have solved problems rotating frames. Here the basic result is as follows. For a particle moves on a circle, of radius R, with constant angular velocity ω. The acceleration of the particle has magnitude $\omega^2 R = v^2/R$

[Q/T]:= I would like to know if you have seen the derivation of the above result on acceleration of a particle moving on a circle. Have you seen the proof? What type of proof was given to you in your earlier classes?;
[TalkTime]:= 5 mins.

It appears that most of you have either not seen the proof or do not recall how the result is derived. Firts of all I want you to clearly understand that *the motion of a particle on a circle has accelereation and it is not uniform motion. Any one write uniform motion again, will get negative marks. If the particle moves on a circle of radius R with constant angular velocity ω the acceleration is $\omega^2 R$ and is also equal to v^2/R. The direction of the acceleration is towards the centre of the circle.*

I will give you two simple proofs. One will be geometric proof and the other one will be analytical.

Proof-1: Let θ be the angle made by the position vector of the particle at time t with some fixed initial direction. Then $\theta = \omega t$. If $\vec{x}(t)$ is the position of the particle at time t, then to compute the the acceleration we go back to its definition as rate of change of velocity

$$\begin{aligned} \vec{a} &= \frac{d^2\vec{x}(t)}{dt^2} = \frac{d\vec{v}(t)}{dt}. \\ &= \lim_{\Delta\to 0}\frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t}. \end{aligned} \tag{17.2}$$

Let the points P, Q denote the positions of the particle at times t and $t+\Delta t$. Since the direction of the velocity at any point is along the tangent and the speed is constant, the velocity at the two point P, Q has the directions as shown in Fig.**??**. To compute the difference needed in Eq.(17.1)

Let $\Delta\theta = \omega\Delta t$ be the angle turned in time Δt. To geometrically find the change in velocity in time Δt we draw the vector $\overrightarrow{QC}$ parallel to the velocity vector at time $\overrightarrow{PA}$ then

$$\begin{aligned} \vec{v}(t+\Delta t) - \vec{v}(t) &= \overrightarrow{QB} - \overrightarrow{QC} & (17.3) \\ &= \overrightarrow{CB}. & (17.4) \end{aligned}$$

The magnitude of the change in velocity is CB and is equal to the length of $QB = QC$ multiplied by the angle $\angle BQC = QC\,\Delta\theta = v\,\omega\,\Delta t$.

$$|\vec{v}(t+\Delta t) - \vec{v}(t)| = v\omega\Delta t \qquad (17.5)$$

Therefore, from Eq.(17.1) we get

$$\begin{aligned} \therefore \qquad a &= \lim_{\Delta\to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t} & (17.6) \\ \text{or} \qquad a &= \omega v = \frac{v^2}{R} & (17.7) \end{aligned}$$

This gives the amplitude of the acceleration.

Proof-1: For a particle rotating on a circle the x_1, x_2 coordinates are given by

$$\begin{aligned} \vec{x} &= R\cos\theta\,\hat{e}_1 + R\sin\theta\,\hat{e}_2 & (17.8) \\ &= R\cos\omega t\,\hat{e}_1 + R\sin\omega t\,\hat{e}_2. & (17.9) \end{aligned}$$

Differentiating this above relation w.r.t. time we get

$$\begin{aligned} \vec{v} = \frac{d\vec{x}}{dt} &= \frac{d}{dt}[R\cos\omega t\hat{e}_1 + R\sin\omega t\hat{e}_2] & (17.10) \\ &= (-R\omega\sin\omega t\,\hat{e}_1 + R\omega\cos\omega t\,\hat{e}_2). & (17.11) \end{aligned}$$

Therefore, differentiating once again

$$\begin{aligned} \vec{a} &= \frac{d\vec{v}}{dt} \\ &= (-R\omega^2\cos\omega t\,\hat{e}_1 - R\omega^2\sin\omega t\,\hat{e}_2) & (17.12) \\ &= \omega^2(R\cos\omega t\,\hat{e}_1 + R\sin\omega t\,\hat{e}_2) & (17.13) \\ &= -\omega^2 R\vec{x}. & (17.14) \end{aligned}$$

For motion on a circle the magnitude of the acceleration is therefore given by $a = \omega^2 R$ and is directed opposite to the position vector $\vec{x}$ and hence the acceleration is towards the centre.

The angular velocity is a vector quantity with magnitude ω and direction given by $\hat{e}_3$ which is just the axis of rotation, so that $\vec{\omega} = \omega\,\hat{e}_3$.

I will write the velocity in a slightly different form. Taknig the cross product $\hat{e}_3 \times \vec{x}$, it is now easily verified that the velocity of the particle can be written as

$$\vec{v} = \hat{e}_3(\omega \times \vec{x}) = \vec{\omega} \times \vec{x}. \qquad (17.15)$$

I now need to derive similar results in more general setting. This is because we want to discuss three important applications.

1. The first one you have already seen. This is a body moving on a circle with constant angular velocity.

2. The second application will be writing equations of motion for a particle moving under action of some forces. You are viewing it from a noninertial frame which is rotating with angular velocity which may not be constant in time.

3. The third application, slightly different from what has been listed above, is the motion of a rigid body as seen from an inertial frame. This will be the last topic in this course.

I will derive now an expression for acceleration of a moving particle, as seen from a rotating frame, by differentiating the formula Eq.(17.1). This what I started doing last time and have not given details of all the steps. Once we are able to write the relation between accelerations as seen from an ainertial frame and as seen from a rotating frame, we will be able to write the EOM for the particle and anlyze it from the point of view of an observer in a rotating frame. One of the applications will to Foucault pendulum to understand why the plane of oscillations rotates.

I will derive a slighlty general result than what is needed immediately and therefore I will deviate from the proposed outline of differentiating the Eq.(17.1) twice to obtain the desired realation between accelerations.

In fact, right now I want to go back to early lectures in this course where it was asked what is a vector quantity.

§3 Definition of Vectors in Terms of Rotations

§3.1 First Ideas about Vectors

The vectors are frequently introduced as objects having magnitude and direction. You are familiar with this definition. This definition is not very precise and has limited usefulness.

To drive the point home, my friend KPN Murthy would like to ask if an elephant moving towards north is a vector quantity?. What is your answer? Is it a vector or not?

This approach has its limitations. Specially when we want to get answers as numbers and compare with experiments, we must work with the components of position and vectors.

§3.2 An Abstract Vector Space

We at first encounter vectors in three dimensions. They can be added and can be multiplied by real numbers and the result of these oerations is again a vector. These operations on vectors satisfy well defined properties and a carefully selected set of transformations and properties can be taken as abstract definitions of vectors.

The mathematicians keep doing abstractions in this way. They have a concept defined in a particular context and a set of theorems are proved. A set of results, theorem(s), are then taken as defining properties of the concept.

A list of the properties which can be used as defining vectors has been given in an earlier lecture and goes beyond the usual vectors in three dimensions. This has led to study of vector spaces as an object of interest by themselves and to inclusion of a large variety of mathematical objects being treated as vectors. All the result derived in general setting will then be applicable to any object having the properties required of vectors.

§3.3 Vectors as Objects with Three Components

The vectors in 3 dimensions are also realized by writing them as linear combinations of unit vectors along the three coordinate axes. Thus ordinary vectors, in 3 dimensions, can also be concieved as objects with three components with component wise addition and multiplication by real numbers.

We have seen that the components of position vector along three coordinate axes change under a rotation of axes and the new components are given by

$$\vec{x}' = \vec{x} - \sin\alpha(\hat{n} \times \vec{x}) + (1 - \cos\alpha)\hat{n} \times (\hat{n} \times \vec{x}). \tag{17.16}$$

The components of the position vector of a point P are conveniently computed by taking scalar products of the position vector $\overrightarrow{OP}$ with the unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ along the axis. Thus

$$x_k = \hat{e_3} \cdot \overrightarrow{OP}. \tag{17.17}$$

This relation between the components will apply to every vector quantity such as force, velocity, acceleration and electric field.

Our discussion on transformation of components of vectors under rotation of axes open up the definition of vectors as objects with three components whose components along the coordinate axes have the same relation as has been derived for the position vector of a point. *You may think of a vector $\vec{A}$ as an object with three components (A_1, A_2, A_3) which under a rotation transform in the same manner as position coordinates in Eq.(17.1).*

$$\vec{A}' = \vec{A} - \sin\alpha(\hat{n} \times \vec{A}) + (1 - \cos\alpha)\,\hat{n} \times (\hat{n} \times \vec{A}). \tag{17.18}$$

[R/S]:= You are calling vector as something that statisfies this condition?

[A/S]:= We require this property for vectors, do not call this a condition to be statisfied by vectors.

[Q/S]:= But why are you now taking this as definition of vectors? You have already defined vectors, so?

I suppose your question is "Why make life more complicated and cause confusion by introducing one more definition?"

[Q/S]:= The older definition is sufficient for many purposes. There will new situation when one needs to generalize the defintion of vectors. This will be the case when you learn about tensors. You cannot think about tensors in older language. You cannot think of a second rank tensor as an object with magnitude and direction.

But you can generalize the cocept of vectors to tensor in this new language.

Therefore, my answer in one sentence is that this is the definition that allows us to generalise the concept of vectors to tensors.

This process is very common in mathemtics. You start with some definition of a new concept, derive some properties and then take one of them as defining property of the concept you started with. This is how you generalise and go to a situation where the new definition will be applicable to a larger class.On the other hand older definition will not be applicable to a larger class, and will be applicable to only in a restriced context.

[Q/S]:= But how do you decide which definition to generalize?

[A/S]:= The mathematicians have their own intuition to decide. It is difficult to tell how they decide which property is useful definition to generalise.

We actually encounter tensors in several areas of Physics. The moment of inertia that you have already heard about is a tensor of second rank and we will have a chance to talk more about it in the last part of this course.

[R/A]:= A vector we can think of a vector as an array and generalize it to a second rank tensor as a matrix and so on.

[R/T]:= Yes, you can represent a tensor by a matrix. But a tensor is more than compom-nents written as a matrix. The components must transform in a well defined way. The tensor is a whole object and this or that element of a matrix. In fact you can get rid of components and still talk about tensors without any reference to coordinate axes.

§4 Time Derivative of a Vector in a Rotating frame

The velocity and acceleration of a point particle can be computed by taking the first and second derivative of the position vector w.r.t. time. I will derive this result and propose to do this for a more general situation.

Let us consider a vector that does not appear in mechanics. This will help you in fixing your ideas. So consider example of an electric field which is changing with time as seen by an observer in the first frame K. The components of the electric field will be obtained by taking the dot product of the electric field with unit vectors along the axes. So

$$E_k = \vec{E} \cdot \hat{e}_k, = |\vec{E}| \cos \alpha_1, \qquad k = 1, 2, 3. \tag{17.19}$$

where α_k is the angle bewteen the electric field and the axis X_k, (k=1,2,3). This process of getting components is the same as for the position vector. So the components of electric field, (or of every other vector) will transform in the same way as the components of position vector of a point. Thus as far as the components of the field itself in the two frames are concerned the relation between them will be the same as for the position vector.

Not only I need to know how components in the two frames are related but I also want to know what are the components of its first and second derivatives of the electric field, as seen by an observer in a rotating frame. This requires some computation.

Let (A_1, A_2, A_3) be components of a vector $\vec{A}(t)$ as seen by an observer in a frame K and $(A'_1, A'_2, A_3)'$ be the components of the same vector as seen by another observer in frame K'. The vector may be changing with time. I want to relate the first and the second time derivatives of the components of $\vec{A}$ as seen by the two observers.

§5 Gallilean Transformations

The equation of motion

$$m\vec{a} = \vec{F} \tag{17.20}$$

has the same form in all inertial frames. If we rotate the axes, the components of acceleration and force will be different as seen by different observers using rotated axes, but the form of equations does not change. This happens because both sides of the equation of motion transform in the same way when the axes are rotated.

Similarly, the equations of motion do not change if the origin of the axes K' is translated to a some other fixed point.

The set of all such operations, shifting the origin, rotating the axes and changing to another inertial frame and their combinations are known as *Gallilean transformations* and we say that the Newton's laws are do change under Gallilean transformations.

§6 Time Derivative of a Vector in a Rotating Frame

I have a particle moving under action of external forces. I want to describe the motion of the particle as seen by an observer in a rotating frame K'. The uderstanding is that the frame is rotating w.r.t. an inertial frame K. For example, you are watching the Focault pendulum sitting on the earth which is spinning baout its own axis. For the duration of time the observations are being made the orbital motion of the earth may be neglected. For this purpose K is the frame fixed in stars and K' is the frame fixed in the earth.

I will assume that the coordinate axes have been chosen in such a way that the third axes, OX_3 is just the axis of rotation. For every vector $\mathbf{A}$, the relation between the components, $\vec{A} = (A_1, A2, A_3)$ and $\vec{A}' = (A'_1, A2', A'_3)$ as seen by the two observers, is given by

$$A'_3(t) = A_3(t) \tag{17.21}$$

$$A'_1(t) = A_1(t)\cos(\omega t) + A_2(t)\sin(\omega t) \tag{17.22}$$

$$A'_2(t) = -A_1(t)\sin(\omega t) + A_2(t)\cos(\omega t) \tag{17.23}$$

We compute the time derivative of the components of the vector, assuming it to be time dependent. Differentiating $A'_1(t)$ and $A'_2(t)$ w.r.t. time we getting

$$\begin{aligned}
\frac{dA'_1(t)}{dt} &= \frac{dA_1(t)}{dt}\cos(\omega t) + A_1(t)(-\omega)\sin(\omega t) + && (17.24)\\
&\qquad \frac{dA_2(t)}{dt}\sin(\omega t) + A_2(t)\omega\cos(\omega t) && (17.25)\\
&= \frac{dA_1(t)}{dt}\cos(\omega t) + \frac{dA_2(t)}{dt}\sin(\omega t)) && (17.26)\\
&\qquad -\omega\Big(A_1(t)\sin(\omega t) + A_2(t)\cos(\omega t)\Big) && (17.27)\\
\frac{dA'_2(t)}{dt} &= -\frac{dA_1(t)}{dt}\sin(\omega t) - \omega A_1(t)\cos(\omega t) + \\
&\qquad \frac{dA_2(t)}{dt}\cos(\omega t) - \omega)A_2(t)(\sin(\omega t) && (17.28)\\
&= -\frac{dA_1(t)}{dt}\sin(\omega t) + \frac{dA_2(t)}{dt}\cos(\omega t) && (17.29)\\
&\qquad -\omega\Big(A_1(t)\cos(\omega t) + A_2(t)(\sin(\omega t))\Big) && (17.30)
\end{aligned}$$

The above results can be written as

$$\begin{aligned}
\left(\frac{d\vec{A}'}{dt}\right)_{\rm rf} &= \left(\frac{d\vec{A}}{dt}\right)_{\rm inf} + \vec{\omega}\times\vec{A} && (17.31)\\
\left(\frac{d\vec{A}}{dt}\right)_{\rm inf} &= \left(\frac{d\vec{A}'}{dt}\right)_{\rm rf} - \vec{\omega}\times\vec{A}'. && (17.32)
\end{aligned}$$

Here suffixes "inf" and "rf" stand for inertial frame and rotating frame, respectively. The last equation follows from the previous one by an exchange of $\vec{A}$ with $\vec{A}'$ and changing the sign of ω.

Lesson 18

Coriolis Force Examples

Sept 21, 2012

§1 EOM in A Rotating Frame

Today I continue my discussion of how to write EOM in an non inertial frame. The non inertial frames of reference of interest is a rotating frame. I will discuss some of the consequences of working in a rotating frame. We have built up the formalism for this discussion and today I hope to complete it and apply to some well know cases.

Let me remind you of what we did last time. In the process of building up transformations under rotation of coordinate axes, I have tried to seamlessly integrate some other topics. For example, we have now arrived at a definition of vectors as an object which has three components transforming under rotations in the same manner as the components of position vector of a point. Thus every vector $\vec{A}$ has three components transforming as

$$\vec{A} = \vec{A}' - \sin\theta\,(\hat{n}\times\vec{A}) + (1-\cos\theta)\,\hat{n}\times(\hat{n}\times\vec{A}) \tag{18.1}$$

under rotations.

We have been concerned with relation between the rates of change of vector as seen from an inertial frame and a rotating, non inertial, frame.

Last time I derived an equation relating the rate of change of a components of a vector in inertial and rotating frames. This relation, for an arbitrary vector, is given by

$$\left(\frac{d\vec{A}'}{dt}\right)_{\text{rf}} = \left(\frac{d\vec{A}}{dt}\right)_{\text{inf}} + \vec{\omega}\times\vec{A} \tag{18.2}$$

$$\left(\frac{d\vec{A}}{dt}\right)_{\text{inf}} = \left(\frac{d\vec{A}'}{dt}\right)_{\text{rf}} - \vec{\omega}\times\vec{A}'. \tag{18.3}$$

Here suffixes "inf" and "rf" stand for rotating frame and inertial frame, respectively. This equation Eq.(18.2) was obtained by differentiating Eq.(18.1) and writing $\frac{d\theta}{dt} = \omega$ and some algebraic manipulations were needed to get the answers in the final form Eq.(18.3). This rule of relating time derivatives in the two frames is usually written in the form

$$\left.\frac{d}{dt}\right|_{\rm rf} = \left.\frac{d}{dt}\right|_{\rm inf} - \vec{\omega}\times . \tag{18.4}$$

with the understanding that the both sides act on a vector quantity. The rate of change of vector components as seen from a rotating frame consists of two parts. The first part comes from the fact that vector itself is changing and the second part reflects the time dependence of the components coming from the fact that the coordinate axes are rotating.

In order to be able to use Newton's Laws in an inertial frame to write the EOM of a particle as seen by an observer sitting in a non inertial frame, we need to relate the second order time derivatives. This is now straight forward.

$$\left.\frac{d}{dt}\left(\frac{d\vec{A}}{dt}\right)_{\rm in}\right|_{\rm in} = \frac{d}{dt}\left(\left(\frac{d\vec{A}'}{dt}\right)_{\rm rf} - \vec{\omega}\times\vec{A}'\right)_{\rm rf} \tag{18.5}$$

$$= \left(\frac{d}{dt} - \vec{\omega}\times\right)_{\rm rf}\left(\left(\frac{d\vec{A}'}{dt}\right)_{\rm rf} - \vec{\omega}\times\vec{A}'\right)_{\rm rf} \tag{18.6}$$

$$= \frac{d^2\vec{A}}{dt^2} - \dot{\vec{\omega}}\times\vec{A}' - 2\vec{\omega}\times\frac{d\vec{A}'}{dt} + \vec{\omega}\times(\vec{\omega}\times\vec{A}') \tag{18.7}$$

If $\vec{x}'$ and $\vec{v}'$ denote the position and velocity of a particle as seen from a non inertial frame, then accelerations in the two frames are related by

$$\left.\frac{d^2\vec{x}}{dt^2}\right|_{\rm inf} = \left.\frac{d^2\vec{x}'}{dt^2}\right|_{\rm inf} - \dot{\vec{\omega}}\times\vec{x}' - 2\vec{\omega}\times\frac{d\vec{x}'}{dt} + \vec{\omega}\times(\vec{\omega}\times\vec{x}'), \tag{18.8}$$

$$= \left.\frac{d^2\vec{x}'}{dt^2}\right|_{\rm inf} - \dot{\vec{\omega}}\times\vec{x}' - 2\vec{\omega}\times\vec{v}'(t) + \vec{\omega}\times(\vec{\omega}\times\vec{x}'). \tag{18.9}$$

If $\vec{F}$ is the net force acting on the particle, the mass times acceleration in the inertial frame, the left hand side of Eq.(18.9), must be equal to the force. This gives the EOM in the non inertial frame as

$$m\left.\frac{d^2\vec{x}'}{dt^2}\right|_{\rm rf} = \vec{F} + m\dot{\vec{\omega}}\times\vec{x}' + 2m\vec{\omega}\times\vec{v}'(t) - m\vec{\omega}\times(\vec{\omega}\times\vec{x}'). \tag{18.10}$$

When the frame rotates with a constant angular velocity, $\dot{\omega} = 0$, and one has the EOM in a rotating frame given by

$$\left.\frac{d^2\vec{x}'}{dt^2}\right|_{\rm rf} = \frac{\vec{F}}{m} + 2\vec{\omega}\times\vec{v}' - \vec{\omega}\times(\vec{\omega}\times\vec{x}'). \tag{18.11}$$

The second and the third terms are the pseudo forces that are to be added to the force in order to correctly describe the motion of a particle in a rotating frame. The term $2\vec{\omega}\times\vec{v}'$ is known as the Coriolis force and the term $\vec{\omega}\times(\vec{\omega}\times\vec{x}')$ is the well known as the centrifugal force term.

Now onwards we will now drop the superscript ′ and use $\vec{x}$ to denote the position of the particle as seen from a rotating frame.

§2 Effects of Coriolis Force

§2.1 Foucault Pendulum

The position as seen in an inertial frame will not appear in our discussions below. Hence we will now drop the superscript $'$ and use $\vec{x}$ to denote the position of the particle as seen from a rotating frame and write the EOM as

$$\left.\frac{d^2\vec{x}}{dt^2}\right|_{\rm rf} = \frac{\vec{F}}{m} + 2\vec{\omega}\times\vec{v} - \vec{\omega}\times(\vec{\omega}\times\vec{x}). \tag{18.12}$$

Consider a point P on the earth at latitude λ. Choose the X_1 axis towards east, X_2 towards north and X_3 axis vertically upwards as shown in Fig.19 below. The angular velocity vector of the earth lies in a plane containing the axis of spin of the earth and the X_3 axis. It has horizontal component $\omega\cos\lambda$ along the X_2 axis and a vertical $\omega\sin\lambda$.

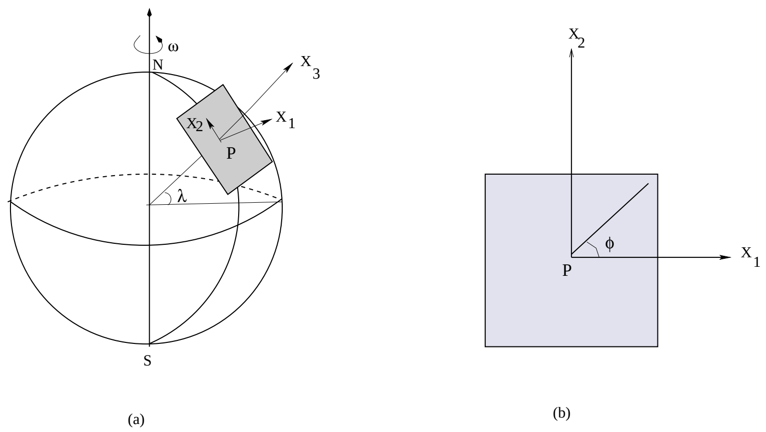

Fig. **19.**

The angular velocity vector is therefore given by

$$\vec{\omega} = (0, \omega\cos\lambda, \omega\sin\lambda). \tag{18.13}$$

Therefore,

$$\vec{\omega}\times\vec{v} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ 0 & \omega\cos\lambda & \omega\sin\lambda \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{18.14}$$

$$= (\cos\lambda v_3 - \sin\lambda v_2)\omega\hat{e}_1 + v_1\omega\sin\lambda\hat{e}_2 - v_1\omega\cos\lambda\hat{e}_3 \tag{18.15}$$

Neglecting the vertical motion of the pendulum, the EOM for the motion in the X_1, X_2

plane are given by

$$\ddot{x}_1 = -v_2\omega\sin\lambda = -\dot{x}_2\omega\sin\lambda \tag{18.16}$$
$$\ddot{x}_2 = v_1\omega\sin\lambda = \dot{x}_1\omega\sin\lambda \tag{18.17}$$

Multiply (18.16) by x_2 and (Eq.(18.17)) by x_1 to get

$$x_2\ddot{x}_1 = -x2\dot{x}_2\omega\sin\lambda, \tag{18.18}$$
$$x_1\ddot{x}_2 = x_1\dot{x}_1\omega\sin\lambda. \tag{18.19}$$

Subtracting the above two equations we get

$$x_2\ddot{x}_1 - x_1\ddot{x}_2 = -(x_2\dot{x}_2\omega + x_1\dot{x}_1)\omega\sin\lambda \tag{18.20}$$
$$\text{or} \qquad \frac{d}{dt}(x_2\dot{x}_1 - x_1\dot{x}_2) = -\frac{d}{dt}(x_1^2 + x_2^2)\omega\sin\lambda \tag{18.21}$$

Introducing polar coordinates ρ, ϕ by means of equation

$$x_1 = \rho\cos\phi, \quad x_2 = \rho\sin\phi, \tag{18.22}$$

we note that

$$x_2\dot{x}_1 - x_1\dot{x}_2 = \rho^2\frac{d\phi}{dt} \tag{18.23}$$

which allows us to write Eq.(18.21) in the form

$$\frac{d(\rho^2\phi)}{dt} = -\Big(\frac{d\rho^2}{dt}\Big)\omega\sin\lambda, \tag{18.24}$$
$$\therefore \qquad \rho^2\frac{d\phi}{dt} = -\rho^2\omega\sin\lambda, \tag{18.25}$$
$$\Rightarrow \qquad \frac{d\phi}{dt} = -\omega\sin\lambda. \tag{18.26}$$

Therefore, the plane of the pendulum rotates with angular velocity $\omega\sin\lambda$

§2.2 Sideways deflection of freely falling body

A body is dropped from height h just above the point P, at time $t = 0$ it will get a small deflection sideways and not reach ground at the point P. We will now calculate this sideways deflection. The x_1 component of equation of motion Eq.(18.12) is

$$m\frac{d^2x_1}{dt^2} = 2m\omega(\cos\lambda v3 - \sin\lambda v_2) = 2m\omega\cos\lambda gt \tag{18.27}$$

where we have used $v_3(t) = gt$ and dropped v_2 to write the last step in Eq.(18.27). Integrating twice we get

$$\ddot{x}_1 = 2gt(\sin\lambda) \Rightarrow \dot{x}_1 = gt^2\sin\lambda \Rightarrow x_1 = \frac{gt^3}{3}\sin\lambda. \tag{18.28}$$

Substituting the time taken to reach ground from a height h as $t = \sqrt{\frac{2h}{g}}$, the expression for sideways deflection becomes

$$x_1 = \frac{g}{3}\left(\frac{2h}{g}\right)^{3/2} \omega \sin\lambda. \tag{18.29}$$

Lesson 19

Centrifugal Force Examples -II

Sept 28, 2012

§1 Effect of Earth's Rotation

Since the earth is spinning around its own axis, all stationery objects experience centrifugal force on the earth. This causes minute changes in the weight of bodies on the earth. This change varies from point to point on the earth. The centrifugal force is $m\omega^2 R$ where R is the radius of the circle in which the body moves due to earth's rotation. See Fig.20(a) below. The radius R is $R_0 \cos\lambda$, where λ is the latitude of the point where the body is located.

North and South Poles: Consider a body hung with a string at the north pole. Such a body, located at the north or the south pole does not experience any centrifugal force because the radius $R = 0$. There is no change in the force of attraction felt by the body towards the centre of the earth.

Equator The due the attraction of the earth is towards the centre of the earth and the centrifugal (pseudo)force acts in the opposite direction. Thus the force of attraction experienced by the body is $mg - m\omega^2 R_0$ and it acts towards the centre. Apparent weight is given by , see Fig.20

$$W = mg - m\omega^2 R_0. \tag{19.1}$$

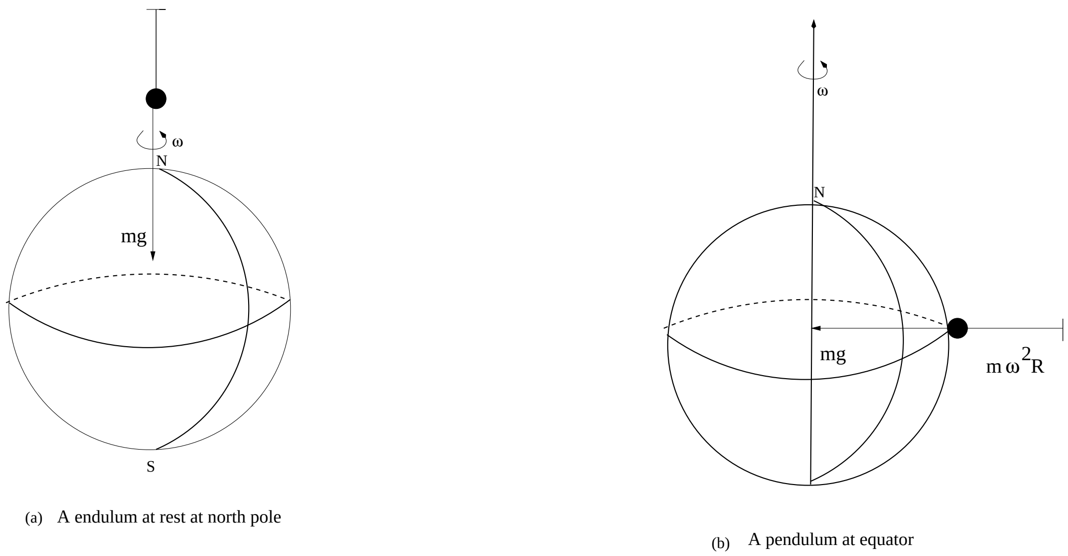

Fig. **20.**

Example 6: The change in weight is maximum at equator and for a body of 10 kg the apparent weight can be computed.(See HCV)

$$\text{Radius of the earth} R_0 = 6400\text{km} = 64 \times 10^5 m \tag{19.2}$$

$$\text{Angular Velocity of the Earth} \omega = \frac{2\pi}{24 \times 60 \times 60} = \tag{19.3}$$

At a Other Points At any point on the earth, other than the poles and on the equator, the direction of the centrifugal (pseudo)force is not in the same line as the gravitational attraction of the earth. While the gravitational attraction is always towards the centre of the earth, the centrifugal force makes and angle $\frac{\pi}{2} + \lambda$ with the line joining the body and the centre of the earth. See Fig.21.

The resultant R of the two forces $\vec{P}, \vec{Q}$ making an angle α is is given by the formula, see Fig.21.

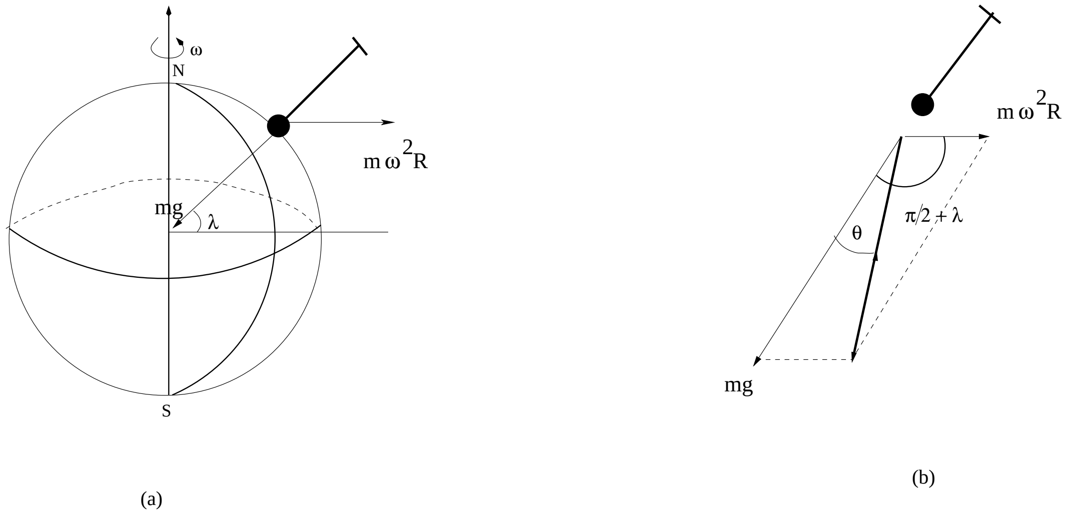

Fig. **21.**

$$R^2 = P^2 + Q^2 + 2PQ\cos\alpha \tag{19.4}$$

$$\tan\theta = \frac{Q\sin\alpha}{P + Q\cos\alpha} \tag{19.5}$$

where R is the magnitude of the resultant force $\vec{R}$ and θ is the angle that $\vec{R}$ makes with $\vec{P}$.

In our case

$$P = mg; \qquad Q = m\omega^2 R; \quad \alpha = \frac{\pi}{2} + \lambda \tag{19.6}$$

Therefore the apparent weight which is the vector sum of the earth's attraction and the centrifugal force is given by

$$W = m\sqrt{g^2 + (\omega^2 R\sin\lambda)^2 - 2g(\omega^2 R\sin\lambda)} \tag{19.7}$$

$$\tan\theta = \frac{\omega^2 R\sin\lambda}{g + \omega^2 R\sin\lambda}, \qquad \cos\alpha = -\sin\lambda \tag{19.8}$$

$$\tag{19.9}$$

where θ is the angle that a suspended body makes with the true vertical, *i.e* with the radius vector drawn from the centre of the earth.

§2 Banking of Railway Tracks and Roads

For a train taking a turn, while moving on a circular track, the centripetal force is provided by the reaction of the railway lines. The magnitude of the reaction needed is mv^2/R. For a fast moving trains and also for a sharp turns will experience to outward centrifugal horizontal force. With only elastic forces due to the track to balance this outward horizontal force in normal course, this will cause a strain in the tracks and for large speeds and sharp curves the railway lines may get damaged. In such a case the train will jump off the tracks. This is avoided by banking of the railway tracks near sharp turns.

The curved part of the railway track not kept in a horizontal plane it is tilted towards the centre of the circular arc for the required movement of the train. The normal reaction due to railway line will then have a vertical and a horizontal components. With suitable banking the centrifugal force and the horizontal components will nearly balance, leaving a much smaller stress on the railway line.

Essentially the same thing works for banking of the roads near a sharp turn. Without banking of the road, frictional forces will provide suitable centripetal force required for turning a vehicle. Since there a limit to maximum possible value of the frictional force, beyond certain speeds and for sharp curves the vehicle will skid and banking of the road is a must for smooth driving.

§3 Conservation Laws

The next set of topics for discussion will be

1. Work Energy Theorem and Conservation of Energy
2. Conservation of Linear Momentum
3. Conservation of Angular Momentum

In our discussion of conservation laws there are two important issues that you must pay attention to.

First issue is to know when will a conservation law hold? For example, for a system with two or more bodies, we would like to know the role of internal forces? To take an example, consider two opposite charges moving under influence of Coulomb attraction.

Q/T:= Is total energy conserved? What is your answer? Do we have conservation of energy, momentum, angular momentum for such systems moving under influence of internal forces only?

Next issue is why do we want to discuss the conservation laws? There are several reasons why the conservation laws are important.

The first issue is why are we discussing conservation laws? Why are they so important?

R/S:= They are related to symmetries. Yes conservation laws are related to symmetries and this goes beyond the scope of the present course. I will come back to it in a moment. Let us first discuss immediate use of conservation laws.

Suppose we did not know conservation laws. What will we do? In order to solve a problem we will have to set up EOM which are typically second order differential equations.

For each particle we will have to write an equation of the form

$$m\ddot{\vec{x}}_\alpha = \vec{F}_\alpha. \tag{19.10}$$

Many a times we can get complete answers using conservation laws without solving any differential equation. Even when we cannot get answers fully, we get partial answers and get a good qualitative idea of possible motion of the system.

The second answer is that we need not worry about several details. A lot of things are independent of details. The conservation laws help us in getting to these quantities fast. For example consider a body going on a roller coaster. A solution to EOM will require for example shape of the roller coaster. However to know the velocity at any point we need not know the shape details. Neglecting friction, the energy conservation directly gives the velocity at any point. One can therefore make general statements which are independent of several details.

R/S:= The conservation laws give a unique solution

R/T:= Newton's Laws will have unique solutions if initial conditions are specified. In other words, if the state of the system is known at initial time, the state is uniquely predicted for later times. There are no problems of non uniqueness of the solution. The concept of state means just this. Given the state, EOM and forces the state is known at all times.

The next important point is relation of conservation laws to symmetries. When you are doing an experiment, you as an observer are free to select any coordinate axes you wish and choose the origin where ever you want to. Another person may select the origin some where else and may choose the axes to be rotated w.r.t. the axes you have chosen. The laws of physics are invariant are the same for both persons. The EOM for both observers will be related and the final answers will be the same. If the forces on the system also respect the symmetries, in well defined sense, then we have conservation laws. Many important conservation laws have an intimate relation with symmetries and arise in this fashion.

The usefulness of conservation laws conservation laws goes beyond what has been said so far. When you have system of particles whose interactions are unknown, the symmetries become the guiding principle. Historically physicists have discovered conservation laws by means of experiments and have tried to explore their relation with possible symmetries. Important discoveries have been made in this fashion. The symmetries have then become guiding principle to investigate the properties of unknown interactions and ultimately have led to important discoveries.

When a large number of particles, starting with neutron, were being discovered during the thirties to sixties in the $2)^{\text{th}}$ century, the new symmetries were discovered and found important for modelling the interaction.

For example, experimental results on neutron proton forces being charge independent has led to "isospin" conservation and a symmetry known as $SU(2)$ symmetry.

Relativistic invariance continues to be an important example which has been used to derive general results and to get an understanding of nature of forces.

The gauge theories, and the Higgs boson, that are being talked about these days, were postulated and the unified model of electromagnetic interactions was given based on symmetry considerations. The discovery of Higgs boson is a triumph of this approach of using symmetries to find the unknown.

To summarise:

1. Conservation laws help us in solving the EOM.
2. May conservation laws have been postulated and discovered for systems where forces were unknown.
3. Symmetry arguments are used as useful tools to derive general results valid in a large class of models.
4. Conservation laws allow a qualitative discussion without going into many details. In many such cases a detailed solution may be impossible.
5. The symmetries have been guiding principles in investigation of unknown interactions and have led to new discoveries.

In the next lecture we will begin with our discussion of work energy theorem and energy conservation.

Lesson 20

Work Energy Theorem

Sept 29, 2012

§1 Work Done by a Force

The work done by a force field when a body undergoes an infinitesimal displacement $\Delta\vec{x}$ is defined to be $\vec{F}\cdot\vec{x}$. To compute the work done by force when the body moves on a path C from an initial point I to a final point F is obtained by first dividing the path into a large number of small small parts. The work done for each part is computed and added to give the total work done. Thus

$$W = \sum_k \vec{F}_k \Delta\vec{x}_k. \tag{20.1}$$

Here the sum is over different parts and at the end we take the limit in which the number of parts becomes infinite and the length of each path becomes zero. In this limit the work done is given by an integral

$$W = \int_I^F \vec{F}\cdot d\vec{x} = \int_{t_i}^{t_f} \vec{F}\cdot\frac{d\vec{x}}{dt}dt \tag{20.2}$$

$$= \int_{t_i}^{t_f} (\vec{F}\cdot\vec{v})\ dt. \tag{20.3}$$

Here t_i, t_f denote the initial and final time $\vec{v}$ is the velocity. If θ denotes the angle between work and displacement, $\vec{F}\cdot d\vec{x} = F\cos\theta dx$ and we can write

$$W = \int_I^F \vec{F}\cdot d\vec{x} = \int_i^f F\cos\theta dx. \tag{20.4}$$

In general, the magnitude of the forces and the angle the tangent makes with the force varies from point to point and integration has to be performed to compute the work done.

However in the simple case of magnitude of the work done, as well as the angle θ remain constant, $F\cos\theta$ can be pulled out of the integral and we get

$$W = \int_i^f F\cos\theta dx = F\cos\theta \int_i^f dx = \text{Force} \times \cos\theta \times \text{displacement}. \tag{20.5}$$

which coincides with the definition you would have learnt in your 12th class.

Example 7: Work Done in Circular Motion Assume that a particle moves on a circle with a *constant angular velocity.* The tangential acceleration is zero and the acceleration has only normal component. Hence the force is always towards the centre and normal to the displacement, making $\cos\theta$ factor zero. Thus the total work done in this case between any two points is zero.

Example 8: Work Done in Free Fall When a body falls from height h, freely under influence of gravity, the only force acting on the body is its weight mg. Both the displacement and the force are downwards and hence the angle between them is $\theta = 0$ giving $\cos\theta = 1$. Thus the work done will be mgh.

If the body goes up a height h, the work done will be $-mgh$ because in this case the displacement is opposite to the gravitational force mg and therefore $\theta = \pi$ and $\cos\theta = -1$.

Example 9: Work Done by Gravity for Motion on an Inclined Plane Let us now consider a body sliding on a smooth inclined plane under influence of gravity. If the angle that the plane makes with the horizontal is α, the angle between the gravitational force and the displacement is $\pi/2 - \alpha$ and the work done when the body slides a length ℓ along the plane is

$$W = mg\cos(\pi/2 - \alpha)\ell = mg\ell\sin\theta = mgh. \tag{20.6}$$

where h is the height of the plane.

For a body moving up the plane the angle θ will be $\pi/2 + \alpha$ giving $\cos\theta = \sin\alpha$ and the work done will be $-mgh$.

Example 10: A man lifts a body of mass 5kg resting on the floor and keeps it on a loft at a height of 2m. What is the work done by him on the body.

There are two forces acting on the body. One is the force due to gravity and the other applied by the man. There is no change in kinetic energy and hence work done by the man plus the work done by the gravitational attraction is zero. The work done by the gravitational force is $-mgh = 10g$ J. hence the work done by the man is $+10g$J.

§1.1 Conservative and Non-conservative Forces

An important concept in connection with work is that of conservative force. You must have encountered the concept in your class 12. So I ask you to give me what you know about conservative and non conservative forces.[1]

[1]Reordered, from an earlier lecture

Q/T:= What is conservative force and what is not? Describe the conservative forces in your words.

Your responses along with my comments are summarised in the table below.

Responses of the class on conservative forces

	Response from the class	Comments
[1]	If it is independent of path then it is conservative force	What is independent of path?
[2]	If work done in a path is zero $\Rightarrow$ conservative force	Wrong
[3]	If work done in a loop is zero$\Rightarrow$ In a loop ? Conservative forces	"In **a** loop ?" Which loop? Any one loop?
[4]	Work done depends on initial and final points implies non-conservative force	Wrong
[5]	If potential energy is defined the force is conservative	Agreed. What if potential energy is not defined?
[6]	Work done depends on the path implies conservative force	False
[7]	If work done is independent of path, the force is conservative	OK
[8]	If the total energy remains the same the forces are conservative	Correct. Is reverse true? Does a conservative force imply law of conservation of energy?
[9]	Work done for a conservative force depends only on the displacement and not on the distance	Necessary but not sufficient

Lesson 21

Quiz-III

Oct 3, 2012

First Thirty Minutes for "Quiz-III"

Lecture material repeated and included in other, previous and later lessons.

Lesson 22

Energy Conservation, Examples

Oct 5, 2012

§1 Conservative and Nonconservative Forces...

Definition 1: We have been discussing conservative and nonconservative forces. Recall that *a force is called conservative force if the work done depends only on the end points and does depend on the the paths connecting the two points.* Thus for conservative force we have

$$W(i \to f) = \int_i^f \vec{F} \cdot \overrightarrow{dx} = U(\vec{x}_i) - U(\vec{x}_f), \tag{22.1}$$

where $\vec{x}_i$ and $\vec{x}_f$ are the initial and final positions. The work energy theorem then gives

$$(\text{KE})_f - (\text{KE})_i = U(\vec{x}_i) - U(\vec{x}_f), \tag{22.2}$$

which gives the energy conservation law:

$$(\text{KE})_f + U(\vec{x}_f) = (\text{KE})_i + U(\vec{x}_i), \tag{22.3}$$

It says that *during the motion of a body under action of conservative forces, the total energy, which is a sum of the kinetic energy and the potential energy, remains constant.*

Definition 2: A force field is called conservative if the total work done in a closed path is zero.

We will discuss several examples in one dimension which use conservation of energy.

Clasical region, Turning points

In one dimension a lot of information about the motion can be obtained using energy conservation. So, for examples, the total energy E of a body

$$E = \frac{1}{2}mv^2 + V(x). \tag{22.4}$$

is always greater than or equal to the potential energy $V(x)$. Thus a particle cannot be found in a region where $E < V(x)$. The set of all values of x where $E \geq V(x)$ is called the *classical region*. The boundary points of the classical region are those points where $E = V(x)$. At these point, known as *turning points*. When a body approaches one of these points its velocity becomes zero and the particle will "turn back" to continue motion.

Theorem **3.** *For example for a simple pendulum, the extreme points A, B, where the amplitude is maximum (see Figure) are the turning points. The angle θ at the extreme points takes values $\pm\alpha$ where α is given by*

$$E = mgL(1 - \cos\alpha). \tag{22.5}$$

The allowed values of θ are the values in the range $-\alpha \leq \theta \leq \alpha$ only. The points $\theta = \pm\alpha$ are the turning points of the pendulum.

Lesson 23

Potential Energy of a Spring and Gravitational Field

Oct 8, 2012

Conservative Forces

When a point particle moves from an initial position to some final position under influence of a force field, in general work done depends on the path taken. There are important situations when the work done depends on the initial and final positions of the particle only; *the work done is same for all the paths connecting the two points*. In such a case we say that the force field is **conservative**. In all other cases the force is referred to as the **nonconservative** force. For conservative forces the work done is zero when the initial and the final positions of the particle coincide. An example of nonconservative forces is frictional force. If a particle moves under influence of frictional force, travels some distance and comes back to the original initial position work done is nonzero. Constant force, force due to a spring, electrostatic and gravitational forces furnish examples of conservative forces. For a force field experiments only can decide if it is conservative or nonconservative.

WithoutProof

1. Knowing the expression, $\vec{F}(\vec{x})$ for a force exerted on a particle as function of position of the particle, it is possible to check if the force is conservative or not. It requires use of partial differentiation and will be discussed separately.

2. For conservative forces, we state, *without a proof*, that there exists a function $U(x)$ such that the work done becomes equal to difference in values of U at the two end points.

$$U(\vec{x}_B) - U(\vec{x}_A) = -\int_A^B \vec{F} \cdot dx \qquad (23.1)$$

0 where $\vec{x}_A, \vec{x}_B$ are the position vectors of the two points A, B. This result is a mathematical theorem in vector calculus.

This function U will be called the potential energy. Using the work energy theorem, we can then write the change in kinetic energy as

$$\Delta K = K_B - K_A \tag{23.2}$$

$$= \int_A^B \vec{F} \cdot d\vec{x} = U(\vec{x}_A) - U(\vec{x}_B) \tag{23.3}$$

$$\therefore \quad K_A + U(\vec{x}_A) = K_B + U(\vec{x}_B) \tag{23.4}$$

This is the law of conservation of energy. It states that the sum of kinetic energy and potential energy remains constant. Next we find the potential energy function for a few simple cases.

Constant Force

Potential energy of a particle in a constant force is easily computed using

$$U(x) - U(x_0) = -\int_{x_0}^{x} F dx = -F(x - x_0) \tag{23.5}$$

$$\therefore \quad \boxed{U(x) = -Fx + C} \tag{23.6}$$

where C is a constant, which can be fixed by specifying the value of the potential energy at a reference point. Gravitational force near earth is a example of such a force. Another example of a constant force is the force felt by a charged particle in a uniform electric field.

Potential Energy of a Spring

The work done by forces exerted by a spring are proportional to the change in length of the spring from un-stretched state. If x denotes the extension or compression of the spring, the force exerted by the spring is $F = -kx$ and hence the potential energy is given by

$$U(x) - U(x_0) = -\int {x_0}^x F dx = \int_{x_0}^{x} kx\, dx \tag{23.7}$$

$$= k\Big[\frac{1}{2}x^2\Big]\Big|_{x_0}^{x} \tag{23.8}$$

$$= \frac{1}{2}kx^2 - \frac{1}{2}kx_0^2 \tag{23.9}$$

$$\therefore \quad U(x) = \frac{1}{2}kx^2 + C \tag{23.10}$$

where again C is a constant. Taking the potential energy of the un stretched spring to be zero, $U(0) = 0$, we get $C = 0$ and hence

$$\boxed{U(x) = \frac{1}{2}kx^2} \tag{23.11}$$

Potential Energy of Point Particle in Gravitational Force due to A Sphere

Let us consider a gravitational attraction due to a sphere of mass M on small, point like, body of mass m at some distance r from the sphere. The force of attraction has the magnitude $\frac{GMm}{r} \equiv f(r)$ and is directed towards the centre of the sphere. Taking the centre of the sphere as the origin and $\vec{r}$ as the position vector of the particle the force can be written as

$$\vec{f} = \frac{GMm}{r^2}\frac{\vec{r}}{r} = \frac{GMm\,\vec{r}}{r^3} \tag{23.12}$$

We wish to compute the work done when the particle moves from an initial point A to a final position B. We shall use the fact that the force is conservative, and hence the work can be computed along any conveniently chosen path from A to B. The path we choose consists of two parts AC and CB. The first part AC is an arc of a circle of radius r_A with centre O and the second part CB is a straight line radial path from C to B. The work done along any path Γ will be equal to that along the chosen path running along the arc from A to C and along the straight line segment from C to B. Thus we write,

$$W_{AB} = W_{AC} + W_{CB} \tag{23.13}$$

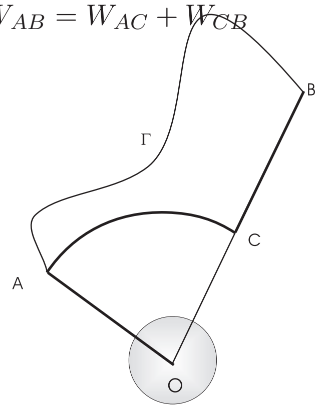

Fig. **22.**

Now the work done along the circular arc is zero because the displacement is perpendicular to the force. When the particle moves from CB the force, being attractive, is towards the origin and is opposite to the displacement from C to B and the angle between the two is π. Hence under a small displacement Δr, the work done is $-f(r)\Delta r$ and the total work done from C to B is

$$W_{CB} = \int_{r_A}^{r_B} f(r)\,dr \tag{23.14}$$

$$= -(GMm)\int_{r_A}^{r_B} \frac{dr}{r^2} \tag{23.15}$$

$$= (GMm)\left[\frac{1}{r}\right]\Big|_{r_A}^{r_B} \tag{23.16}$$

$$= (GMm)\Big(\frac{1}{r_B} - \frac{1}{r_A}\Big) \tag{23.17}$$

If $U(r)$ is the potential energy at a point r, then

$$U(r) - U(r_0) = -\int_{r_0}^{r} f(r)\,dr \tag{23.18}$$

$$= -(GMm)\Big(\frac{1}{r} - \frac{1}{r_0}\Big) \tag{23.19}$$

$$\therefore \quad U(r) = -(GMm)\Big(\frac{1}{r}\Big) + C \tag{23.20}$$

If we select the potential to be zero at infinity, $C = 0$ and one has

$$\boxed{U(r) = -\frac{GMm}{r}} \tag{23.21}$$

for the gravitational potential energy of the particle at a distance r from a sphere of mass M.

An Example: Escape Velocity

As an application of energy conservation, we now obtain expression for escape velocity – the minimum velocity with which a particle is to be projected so that it may escape from the earth.

For a body projected with velocity v from the surface of the earth, the gravitational potential energy of the body on the earth is $-\frac{GMm}{R}$ and total energy will be $E = \frac{1}{2}mv^2 - \frac{GMm}{R}$.

The velocity of projection v will be escape velocity if the body reaches infinity and has zero (minimum) kinetic energy when it reaches there. Note that the gravitational potential energy of the body at infinity is zero.Therefore, the total energy of the body at infinity will be zero. Therefore, using conservation of energy gives

$$\frac{1}{2}mv^2 - \frac{GMm}{R} = 0. \tag{23.22}$$

Thus the escape velocity turns out to be $v = \sqrt{\frac{2GM}{R}}$.

Lesson 24

Collisions and Motion of Centre of Mass

Oct 10, 2012

Let us consider a collision of two particles such that the before and after the collision there are no forces acting on the particle. The only forces that act on the particles are during the collision. These are internal forces and are of action and reaction type. The sum of all such forces is zero and hence in a collision between two particles the total momentum is always conserved.

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2. \tag{24.1}$$

Elastic Collision: A collision is defined to be an *elastic collision* if the total energy is conserved, *i.e.* the total energy after the collision is equal to the total energy before the collision.

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2. \tag{24.2}$$

We will show that, in an elastic collision, the velocity of separation of the particles after the collision equals the velocity of approach of the particles before the collision.

$$u_1 - u_2 = v_2 - V_1. \tag{24.3}$$

Complete Inelastic Collision A collision of two particles is called *inelastic* if after the collision the two particles stick together and move with the same velocity, *i.e.*$(v_1 - V_2 = 0)$.

Most common type of collision is one which is somewhere in between the elastic and inelastic collision. These are characterised by a coefficient of restitution, denoted by e. The **coefficient of restitution** defined by

$$u_1 - u_2 = e(v_2 - V_1). \tag{24.4}$$

and has value between 0 and 1, $0 < e < 1$. $e = 0$ corresponds to totally inelastic collision and $e = 1$ corresponds to an elastic collision.

$$\begin{aligned} m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2. && (24.5)\\ \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 &= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2. && (24.6)\\ & && (24.7) \end{aligned}$$

Rearranging the two equations we get

$$m_1u_1 - m_1v_1 = m_2v_2 - m_2u_2 \quad \Rightarrow \quad m_1(u_1 - v_1) = m_2(v_2 - u_2) \tag{24.8}$$

$$\frac{1}{2}m_1u_1^2 - \frac{1}{2}m_1v_1^2 = \frac{1}{2}m_2v_2^2 - \frac{1}{2}m_2u_2^2 \quad \Rightarrow \quad \frac{1}{2}m_1(u_1^2 - v_1^2) = \frac{1}{2}m_2(v_2^2 - u_2^2). \tag{24.9}$$

Dividing the two equations, we get

$$u_1 + v_1 = v_2 + u_2, \tag{24.10}$$

which gives us the desired result

$$\boxed{u_1 - u_2 = v_2 - v_1.} \tag{24.11}$$

Energy loss in inelastic collision We now show that the energy is not conserved in an inelastic collision and compute the loss of energy. The two particle move with the same velocity V after the collision :$v_1 = v_2 = V$. The momentum conservation implies that

$$m_1u_1 + m_2u_2 = m_1V + m_2V \Rightarrow V = \frac{m_1u_1 + m_2u_2}{m_1 + m_2}. \tag{24.12}$$

$$\begin{aligned} \text{Loss in Kinetic Energy} &= \text{Initial K.E.} - \text{Final kinetic energy} && (24.13)\\ &= [\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2] - [\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2] && (24.14)\\ &= \frac{1}{2}[m_1u_1^2 + m_2u_2^2] - \frac{1}{2}[m_1 + m_2]V^2 && (24.15)\\ & && (24.16) \end{aligned}$$

Substituting for the final velocity from (24.12) we get

$$\begin{aligned} \text{Loss in Kinetic Energy} &= \frac{1}{2}[m_1u_1^2 + m_2u_2^2] - \frac{1}{2}[m_1 + m_2]V^2 && (24.17)\\ &= \frac{1}{2}[m_1u_1^2 + m_2u_2^2] - \frac{1}{2}[m_1 + m_2]\Big(\frac{m_1u_1 + m_2u_2}{m_1 + m_2}\Big)^2 && (24.18)\\ &= \frac{1}{2}\frac{1}{m_1 + m_2}\Big[(m_1u_1^2 + m_2u_2^2)(m_1 + m_2) - (m_1u_1 + m_2u_2)^2\Big] && (24.19)\\ &= \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}\Big((m_1^2u_1^2 + m_2^2u_2^2 + m_1m_2(u_1^2 + u_2^2) - (m_1u_1 + m_2u_2)^2\Big) && (24.20)\\ &= \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}(u_1^2 + u_2^2 - 2u_1u_2). && (24.21) \end{aligned}$$

Introducing the reduced mass $\mu = \frac{m_1 m_2}{m_1+m_2}$, and noting that $(u_1 - u_2)$ is ethe relative velocity before the collision $U_{\rm rel}$, we write the loss in energy as

$$\boxed{\text{Loss in KE} = \frac{1}{2}\mu U_{\rm rel}^2.} \tag{24.22}$$

Go to the top of this lecture click here : 24

Remember, whether the collision is elastic, inelastic or in between, the total momentum is always conserved.

§0.1 Motion of Center of Mass

Let us now consider a system of N particles. Use $m_\alpha, v_\alpha, \vec{p}_\alpha, \vec{f}_\alpha$ to denote their masses, velocities, momenta, and forces on the particles.

The equation of motion for the particles in the system are

$$\frac{d\vec{p}_\alpha}{dt} = \vec{f}_\alpha, \alpha = 1, 2, \cdots, N. \tag{24.23}$$

Summing the above equations over all values of α we get

$$\sum_{\alpha=1}^{N} \frac{d\vec{p}_\alpha}{dt} = \sum_{\alpha=1}^{N} \vec{f}_\alpha \text{ or } \frac{d\vec{P}}{dt} = \vec{F}^{\text{tot}}, \tag{24.24}$$

where $\vec{P}$ is the sum of momenta of all particles in the system and will be called the total momentum of the system.

$$\vec{P} = \sum_\alpha \vec{p}_\alpha. \tag{24.25}$$

and the total force $\vec{F}^{\text{tot}}$ is given by

$$\vec{F} = \sum_\alpha \vec{f}_\alpha = \sum_\alpha \vec{f}_\alpha^{\text{ext}} + \sum_\alpha \vec{f}_\alpha^{\text{int}} \tag{24.26}$$

$$= \sum_\alpha \vec{f}_\alpha^{\text{ext}} \tag{24.27}$$

where we have decomposed the force on each particle into external and internal forces and used the fact that the internal forces cancel pairwise follows from Newton's third law. Thus we have the result that

$$\frac{d\vec{P}}{dt} = \vec{F}^{\text{ext}}. \tag{24.28}$$

Let us look at the left hand side of the above equation. The total momentum is

$$\vec{P} = \sum_\alpha \vec{p}_\alpha = \sum m_\alpha \vec{v}_\alpha \tag{24.29}$$

$$= \sum m_\alpha \frac{d\vec{x}_\alpha}{dt} \tag{24.30}$$

$$= \frac{d}{dt} \sum m_\alpha \vec{x}_\alpha \tag{24.31}$$

Introducing the position of the center of mass by

$$\vec{X}_{\text{cm}} = \frac{\sum m_\alpha \vec{x}_\alpha}{M}, \tag{24.32}$$

we get

$$\vec{P} = M \frac{d\vec{X}_{\text{cm}}}{dt}. \tag{24.33}$$

Here $\vec{X}_{\text{cm}}$ denotes the position of the center of mass defined by and $M \equiv \sum_{\alpha=1}^{N} m_\alpha$ is the total mass of the system. The equation for the position of centre of mass takes the form

$$\frac{d}{dt}\left(M\frac{d\vec{X}_{\text{cm}}}{dt}\right) = \vec{F}_{\text{tot}}^{\text{ext}} \tag{24.34}$$

$$\text{or } M\,\frac{d^2\vec{X}_{\text{cm}}}{dt^2} = \vec{F}_{\text{tot}}^{\text{ext}} \tag{24.35}$$

When the total external force is zero we have the results that

1. The total momentum is conserved,
2. The acceleration of center of mass is zero,
3. The velocity of center of mass remains constant.
4. The center of mass moves like a free particle with a uniform velocity.

When the total external force is not zero, different particles may move in different directions, *but the center of mass always follows the trajectory determined by the total force.* For example, if a bomb is hurled from a plane it will follow a parabolic trajectory. If it explodes before reaching the ground, the fragments may go in different directions with different velocities, but the centre of mass will continue to follow the original trajectory which the bomb would have followed if it had not exploded.

Lesson 25

Examples of Finding Centre of Mass

Oct 12, 2012

Today you will solve a few straightforward simple problems on finding the centre of mass of a few systems consisting of a number of bodies of point particles. Computations in these problems are simple and please open up your notebooks and start working. I will come and check your approach and answers and will start writing out full solutions on the board but after you have made some progress.

1. Three particles of masses 0.5 kg, 1.0 kg and 1.5 kg are placed at the three corners of a right handed triangle of sides 3.0 cm, 4.0 cm and 5.0 cm as shown in the figure. Locate the centre of mass of the system.

   This problem is straightforward application of the definition of the position centre of mass.

2. Three masses m_1, m_2, m_3 are placed on the corners of an equilateral triangle of side $a = 1.0$ m. Find the position of the center of mass of the three particles for $m_1 = 2m, m_2 = 4m, m_3 = 6m$.

3. The pendulum of a clock consists of a heavy rod to which a disk is attached as shown in the figure. Let mass of the rod be 400 gm and that of the disk be 600 gm. The length of the rod is 50 cm and the diameter of the rod is 10 cm. Find the center of mass of the pendulum.

   Note that here we have extended bodies, but their centres of mass are known. In such a situation, for the purpose of finding the centre of mass and we can "pretend" that the entire mass is concentrated at the respective centres of mass and use the same formula to find the centre of mass.

4. A uniform disk of radius has radius $2R$, Find the center of mass of the disk after after a disk of radius R is cut and removed as shown in the figure.

This problem is not a straight forward application of the formula, but you can still do it. So start thinking.

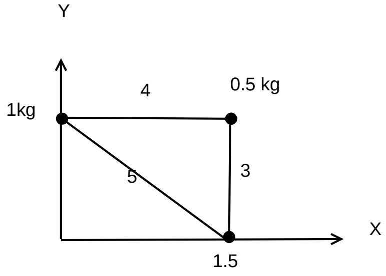

Fig. **23.** for Q1

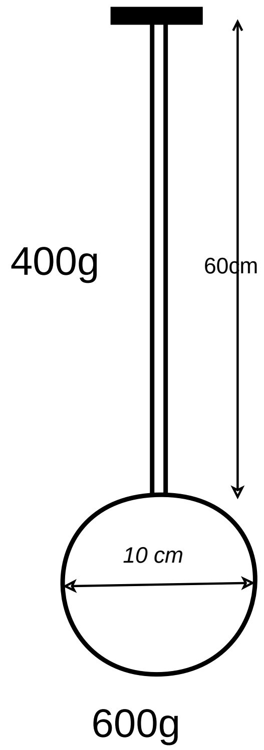

Fig. **24.** for Q3

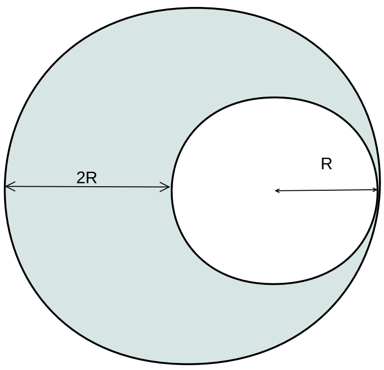

Fig. **25.** for Q4

Lesson 26

Angular Momentum-I: Single Particle

Oct 15, 2012

§1 Introduction

The angular momentum of a particle $\vec{L}$ is defined as

$$\vec{L} = \vec{x} \times \vec{p}. \tag{26.1}$$

For a system of several particles the total angular momentum is sum of angular momenta of individual particles.

An important result for two particle system is that its total angular momentum, (also kinetic energy) can written as sum of two parts. One part refers to the angular momentum (kinetic energy) of the center of mass and the other part is like angular momentum (kinetic energy) of a single particle of some mass μ, known as *the reduced mass*, see below.

A similar decomposition of angular momentum and kinetic energy and angular momentum can be given for system of several particles. This decompostion will be needed for applications to rigid body dynamics.

The aim of next few lectures is to discuss condition for conservation of angular momentum. In later lectures consequences of law of angular momentum will be worked out. In particular the second law of Kepler for planetary motion will be seen to be a simple consequence of conservation of angular momentum.

The elliptic orbits of planets around the Sun will be derived using the twin conservation laws of total energy and angular momentum.

§2 Rate of change of angular momentum single particle

In order to derive the conditions for conservation of angular momentum, we compute the rate of change of angular momentum moving under influence of force $\vec{F}$.

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}((\vec{x}) \times \vec{F} \tag{26.2}$$

$$= \frac{d\vec{x}}{dt} \times \vec{p} + \vec{x} \times \frac{d\vec{p}}{dt}. \tag{26.3}$$

Now use EOM, $\dfrac{d\vec{p}}{dt} = \vec{F}_{\text{tot}}$, to get

$$\frac{d\vec{L}}{dt} = \frac{d\vec{x}}{dt} \times \vec{p} + \vec{x} \times \frac{d\vec{p}}{dt} \tag{26.4}$$

$$= \vec{v} \times (m\vec{v}) + \vec{x} \times \vec{F}_{\text{tot}} \tag{26.5}$$

$$= \vec{\tau} \qquad (\because \vec{v} \times (m\vec{v}) = 0). \tag{26.6}$$

Here $\vec{\tau} = \vec{x} \times \vec{F}_{\text{tot}}$ is torque of the total force acting on the particle. Thus we have important result that the rate of change of angular momentum is given by

$$\boxed{\frac{d\vec{L}}{dt} = \vec{\tau}} \tag{26.7}$$

Thus the angular momentum of a particle is a constant of motion if and only if the particle moves under influence of force which has zero torque.

$$\vec{\tau} = \vec{x} \times \vec{F}_{\text{total}} = 0 \Rightarrow \frac{d\vec{L}}{dt} = 0.. \tag{26.8}$$

Example 11: The conservation of angular momentum means the torque $\vec{r} \times \vec{F}$ is zero. This implies that

(a) $\vec{x}$ is zero,

(b) $\vec{F}$ is zero, or

(c) $\vec{F}$ and $\vec{r}$ are parallel.

Example 12: When $\vec{F} = 0$ the particle moves on a straight line with a constant speed, the value of the angular momentum remains $\vec{x} \times \vec{p}$ the same at all times even though the position vector keeps changing. Verify this yourself explicitly, for two cases (i) the origin lies on the trajectory. (ii) the trajectory does not pass through the origin.

Example 13: For motion of the planets around the Sun the force on a planet is always, directed towards the Sun. Therefore with the Sun taken as the origin, the force $\vec{F}$ and the position vector $\vec{x}$ are always parallel, therefore the angular momentum of the planet is a constant of motion.

Food for your thought: Remember that both torque and angular momentum values change when the choice of origin is changed. So what happens to the law of angular momentum conservation when the origin is shifted to a point diffrent form the Sun?

Example 14: Angluar momentum of a point particle moving in plane
For a particle moving in a plane, taking the origin to lie in the plane, we write the position vector as $\vec{x} = (x_1, x_2, 0)$, where

$$x_1 = r\cos\theta, \qquad x_2 = r\sin\theta.. \tag{26.9}$$

The velocities are, therefore, given by

$$\begin{aligned} v_1 &= \frac{dx_1}{dt} = \dot{r}\cos\theta - r\sin\theta\dot{\theta} & (26.10) \\ v_2 &= \frac{dx_2}{dt} = \dot{r}\sin\theta + r\cos\theta\dot{\theta}. & (26.11) \end{aligned}$$

Only the third component of the angular momentum is nonzero and is given by

$$\begin{aligned} L_3 &= m(\vec{x}\times\vec{v})_3 = m(x_1v_2 - x_2v_1) \\ &= m[r\cos\theta(\dot{r}\sin\theta + r\cos\theta\dot{\theta}) - r\sin\theta(\dot{r}\cos\theta - r\sin\theta\dot{\theta})] \\ &= mr^2\dot{theta}(\sin^2\theta + \cos^2\theta) & (26.12) \\ &= mr^2\dot{\theta} & (26.13) \end{aligned}$$

Thus we get

$$\boxed{L_3 = mr^2\dot{\theta}} \tag{26.14}$$

Note that the equation for rate of change of angular momentum is derived from the EOM and is not an independent result. We see this explicitly for simple pendulum.

Example 15: Simple Pendulum For a simple pendulum the motion is restricted to a plane. Choosing the origin of the coordinate system to lie in the plane, both $\vec{x}$ and $\vec{p}$ and hence the angular momentum $\vec{x}\times\overrightarrow{p}$ perpendicular to the plane. Choosing the coordinate axes as shown in the figure, $\vec{L}$ is along positive axis (see figure) and the only the third component is nonzero and

$$\begin{aligned} \vec{L} &= (0, 0, L_3) \\ L_3 &= |\vec{L}| = m|\vec{x}\times\vec{v})| \\ &= m\,|x|\,v = m\,\ell\,(\ell\dot{\theta}) & (26.15) \\ &= m\ell^2\dot{\theta}. & (26.16) \end{aligned}$$

Also the magnitude of the torque of force mg on the pendulum is

$$\begin{aligned} &\text{torque of} mg \text{about the origin} \\ &\qquad = mg \times \text{perpendicular distance} \\ &\qquad = mg\ell\sin\theta & (26.17) \end{aligned}$$

and the direction is along the neagive X_3 axis. The torque of the tension in the string is zero. Hence the total torque is

$$\vec{\tau} = (0, 0, \tau_3), \qquad \tau_3 = -mg\ell \sin\theta. \tag{26.18}$$

Therefore the third component of the EOM, Eq.(26.7), becomes

$$\begin{aligned} \frac{dL_3}{dt} &= -mg\ell \sin\theta \\ \frac{d}{dt} m\ell^2\theta &= -mg\ell \sin\theta \qquad (26.19) \\ & \qquad (26.20) \end{aligned}$$

or

$$\boxed{\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin\theta}. \tag{26.21}$$

This will be recognised as the EOM of the simple pendulum as given by the second. law.

[Q/T:=] Is the angular momentum conserved for a simple pendulum? Consider the values at different. Explain your answer?

[A/S]:= *Sir, the angular momentum becomes zero the extreme positions because the velocity becomes zero.*

[Q/T:=] Make a rough sketch showing variation of the third component of angular momentum against θ. I will come and check your notebooks.

[Q/T:=] What can you say about the conservation of angular momentum for a conical pendulum?

[A/S]:= *The angular momentum will not be conserved because there is a nonzero torque acting on the pendulum.*

[R/T]:= Correct. But, what about the vertical component?

[A/S]:= *Yes it will remain constant, because the torque is in the horizontal plane.*

[Q/T]:= Here Eq.(26.7) gives the same answer as EOM. Will this always be the case for a point mass? Think of different examples.

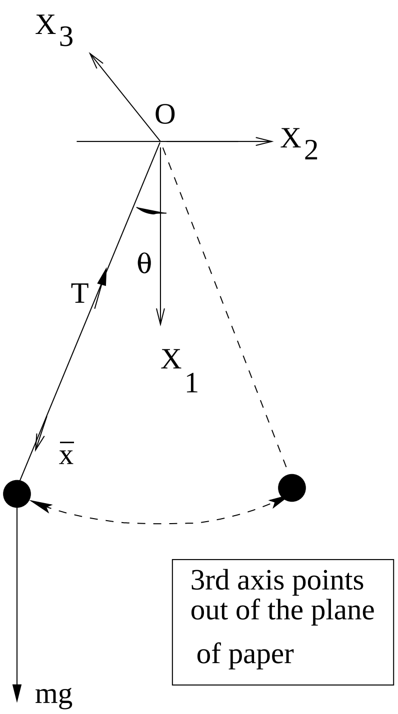

Fig. **26.** Simple Pendulum

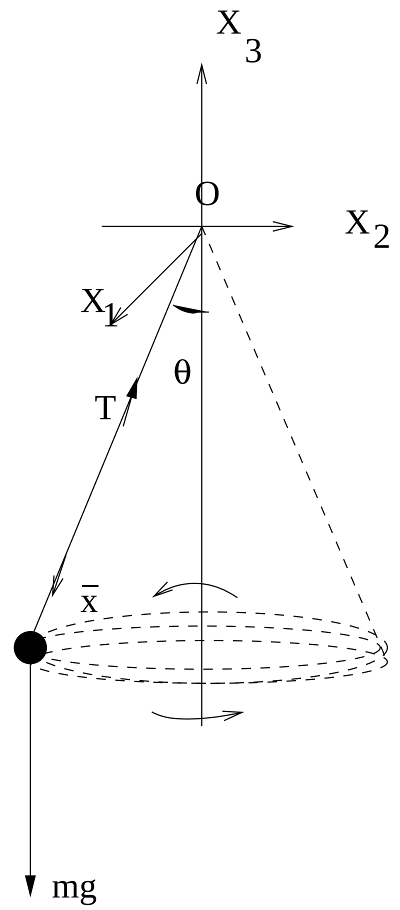

Fig. **27.** Conical pendulum

In a simple example you can see that Eq.(26.7) reduces to $0 = 0$, and does not contain any information.

[Q/S]:= Please give an example where Eq.(26.7) reduces to $0 = 0$.

[A/T]:= *Take a mass connected to a spring and oscillating vertically up and down. Choose the orgin where the other end of the spring is held fixed.*

[Q/S]:= So second law implies the equation for rate of change of angular momentum, but the converse is not true? So the Eq.(26.7) has less information, why even talk about angular momentum? Where does angular momentum become important?

[A/T]:= *Angular momentum equation will be half the set of EOM for rigid body dynamics. For a point particle, like a planet it gives the Kepler's second laws in a very simple and transparent fashion. This is not obvious from the second law EOM.*

Also conservation of angular momentum will hold under very general condions, such as force being central. So the results, following from use of conservation law, will hold for such cases, Such a result does not require all the details about the force law.

Lesson 27

Two Body System: Separation of Center of Mass

Oct 17, 2012

§1 Introduction

An important case of two body system is one in which the two particles interact via a force which depends only on the separation between them. In addition there may be external forces acting on the particles. The Earth system and the hydrogen atom,*i.e.* the electron proton system, are examples interacting via a central force. Our discussion of two body problem will be restricted to those having central forces only.

It will be seen that the EOM for two particle problem decomposes into EOM for the centre of mass and EOM for the relation motion. The EOM for relation motion becomes equivalent to a problem of motion of single particle of reduced mass, (see below). The methods available for one particle problem can be applied to the relative motion.

In the lecture a decomposition of kinetic energy and angular momentum as a sum of quantities referring to the motion of mass and the relative motion is introduced. The EOM also separate into EOM for the centre of mass and for the relative motion. The discussion of the second part, involving relative coordinates, runs very parallel to one body case except that mass should be taken as the "reduced mases" of the system.

The results of this section will be applied to the planetary motion. Similar developments for system consisting of several particles will be given and will be needed for a discussion of rigid body dynamics.

§2 Centre of mass and relative coordinates

Let $\vec{x}_A, \vec{x}_B$ denote the position vectors of two particles A, B, and m_A, m_B be their masses. The velocities of the two particles are given by

$$v_A = \frac{d\vec{x}_A}{dt}, \quad v_A = \frac{d\vec{x}_A}{dt}. \tag{27.1}$$

Let $\vec{X}$ be the position of the centre of mass:

$$\vec{X} = \frac{m_A\vec{x}_A + m_B\vec{x}_B}{m_A + m_B}, \tag{27.2}$$

and the velocity of the centre of mass V is given by

$$\vec{V} = \frac{d\vec{X}}{dt}. \tag{27.3}$$

The vector $\vec{x}$ will denote the relative coordinate

$$\vec{x} = \vec{x}_A - \vec{x}_B \tag{27.4}$$

and the relative velocity is

$$\vec{v} = \frac{d\vec{x}}{dt} = \vec{v}_A - \vec{v}_B. \tag{27.5}$$

It will turn out to be useful to introduce position vectors of the two particles relative to the centre of mass by means of equations

$$\vec{X}_A = \vec{x}_A - \vec{X}, \qquad \vec{X}_B = \vec{x}_B - \vec{X}. \tag{27.6}$$

In turn we have

$$\vec{x}_A = \vec{X}_A + \vec{X}, \qquad \vec{x}_B == \vec{X}_B + \vec{X}. \tag{27.7}$$

It is obvious that

$$\vec{x} = \vec{x}_A - \vec{x}_B = \vec{X}_A - \vec{X}_B. \tag{27.8}$$

Also for later use, we will solve Eq.(27.12) and (27.17) for $\vec{X}_A, \vec{X}_B$. Notice Eqs(27.16) imply

$$\begin{aligned} m_A\vec{X}_A + m_B\vec{X}_B &= m_A\vec{x}_A + m_B\vec{x}_B - (m_A + m_B)\vec{X} \\ &= (m_A + m_B)\vec{X} - (m_A + m_B)\vec{X} \\ &= 0.. \end{aligned} \tag{27.9}$$

Using the above result with Eq.(27.14), $\vec{X}_A - \vec{X}_B = \vec{x}$, we successively get

$$\vec{X}_A = \vec{x} + \vec{X}_B$$

$$\text{and } m_A(\vec{x} + \vec{X}_B) + m_B\vec{X}_B = 0 \qquad [\text{ use } (27.18)]$$

$$\therefore \vec{X}_B = -\frac{m_A}{m_A + m_B}\vec{x} \quad \text{and} \quad \vec{X}_A = +\frac{m_B}{m_A + m_B}\vec{x} \tag{27.10}$$

§3 Equations of motion

The second law gives the EOM for the two particle as

$$m_A \frac{d^2\vec{x}_A}{dt^2} = \vec{F}_A \tag{27.11}$$

$$m_B \frac{d^2\vec{x}_B}{dt^2} = \vec{F}_B. \tag{27.12}$$

Here the total forces acting on the two particles, $\vec{F}_A, \vec{F}_B$ will in general consist of two parts: the external forces $\vec{F}_A^{\rm ext}, \vec{F}_B^{\rm ext}$ and internal forces denoted by $\vec{F}_A^{\rm int}, \vec{F}_B^{\rm int}$.

$$\vec{F}_A = \vec{F}_A^{\rm ext} + \vec{F}_A^{\rm int} \tag{27.13}$$

$$\vec{F}_B = \vec{F}_B^{\rm ext} + \vec{F}_B^{\rm int}. \tag{27.14}$$

As an example keep electron proton system in mind. Here the internal forces are the forces of mutual Coulomb attraction and are always equal and opposite. In addition presence of external electric field will give rise to additional (external) forces. For the internal forces the third law of Newton gives $\vec{F}_A^{\rm int} = -\vec{F}_B^{\rm int}$ and we write

$$\vec{F}_A^{\rm int} = -\vec{F}_B^{\rm int} = \vec{f}. \tag{27.15}$$

With this notation the EOM, (27.11) and (27.12) assume the form

$$m_A \frac{d^2\vec{x}_A}{dt^2} = \vec{F}_A^{\rm ext} + \vec{f} \tag{27.16}$$

$$m_B \frac{d^2\vec{x}_B}{dt^2} = \vec{F}_B^{\rm ext} - \vec{f}. \tag{27.17}$$

Adding these two equations gives

$$\frac{d^2}{dt^2}(m_A\vec{x}_A + m_B\vec{x}_B) = \vec{F}_A^{\rm ext} + \vec{F}_B^{\rm ext}. \tag{27.18}$$

$$\Rightarrow\ M\frac{d^2}{dt^2}\vec{X}_{\rm cm} = \vec{F}_{\rm tot}^{ext}. \tag{27.19}$$

Here $M = m_A + m_B$ is the total mass, $\vec{X}_{\rm cm}$ is the position of the centre of mass of the system. Also $\vec{F}_{\rm tot}^{ext}$ is the sum of all *external forces* acting on the system. The internal forces do not appear in this equation.

§4 Two body system without external forces

Let us now assume that there are no external forces acting on the particle and they interacti via mutual 'action reaction' forces only. In this case $\vec{F}_A^{\rm ext} = \vec{F}_B^{\rm ext} = 0$ and the centre of mass moves like a free particle (see Eq.(27.19)):

$$M\frac{d^2}{dt^2}\vec{X}_{\rm cm} = 0. \tag{27.20}$$

Also in this case there is a further simpification. To see this multiply (27.16) by m_B and (27.17) by m_A and subtract thw two to arrive at the equation

$$m_A m_B \frac{d^2}{dt^2}(\vec{x}_A - \vec{x}_B) = (m_A + m_B)\vec{f}, \tag{27.21}$$

$$\text{or } \mu\frac{d^2\vec{x}}{dt^2} = \vec{f}.. \tag{27.22}$$

Here

$$\vec{x} = (\vec{x}_A - \vec{x}_B) \tag{27.23}$$

denotes relative position and

$$\mu = \frac{m_A m_B}{m_A + m_B} \tag{27.24}$$

is called the *reduced mass* of the two body system. Thus the solution of the two body problem reduces to finding solution to one body problem Eq.(27.27).

§5 Kinetic energy

The kinetic energy of the two particle system is given by

$$\begin{aligned}
\text{KE} &= \frac{1}{2}m_A\dot{\vec{x}}_A^2 + \frac{1}{2}m_B\dot{\vec{x}}_B^2 \\
&= \frac{1}{2}m_A(\dot{\vec{X}}_A + \dot{\vec{X}})^2 + \frac{1}{2}m_B(\dot{\vec{X}}_B + \dot{\vec{x}}_B)^2 \\
&= \frac{1}{2}m_A\dot{\vec{X}}_A^2 + m_A\dot{\vec{X}}_A\cdot\dot{\vec{X}} + \frac{1}{2}m_A\dot{\vec{X}}^2 \\
&\quad + \frac{1}{2}m_B\dot{\vec{X}}_B^2 + \frac{1}{2}m_B\dot{\vec{X}}_B\cdot\dot{\vec{X}}^2 + \frac{1}{2}m_B\dot{\vec{X}}^2 \\
&= \frac{1}{2}\Big(m_A\dot{\vec{X}}_A^2 + m_B\dot{\vec{X}}_B^2\Big) + (m_A\dot{X}_A + m_B\dot{X}_B)\cdot\dot{\vec{X}} + \frac{1}{2}(m_A + m_B)\dot{\vec{X}}^2.
\end{aligned} \tag{27.25}$$

The second term is zero because $(m_A\vec{X}_A + m_B\vec{X}_B) = 0$, see Eq.(27.18). In the remaining two terms we express $\vec{X}_A, \vec{X}_B$ interms of $\vec{x}$ using Eq.(27.19). This gives

$$\begin{aligned}
\text{KE} &= \frac{1}{2}\Big(m_A\dot{\vec{X}}_A^2 + m_B\dot{\vec{X}}_B^2\Big) + +\frac{1}{2}(m_A + m_B)\dot{\vec{X}}^2 \\
&= \frac{1}{2}\Big(\frac{m_A m_B^2}{(m_A+m_B)^2}\dot{\vec{x}}^2\Big)\frac{1}{2}\Big(\frac{m_A^2 m_B}{(m_A+m_B)^2}\dot{\vec{x}}^2\Big) + \frac{1}{2}M\dot{\vec{X}}^2 \\
&= \frac{1}{2}\frac{m_A m_B}{m_A + m_B}^2(m_A\dot{\vec{x}}^2 + m_B\dot{\vec{x}}^2) + \frac{1}{2}M\dot{\vec{X}}^2 \\
&= \frac{1}{2}\frac{m_A m_B}{m_A + m_B}\dot{\vec{x}}^2 + \frac{1}{2}M\dot{\vec{X}}^2
\end{aligned} \tag{27.26}$$

Introducing notation

$$\mu = \frac{m_A m_B}{m_A + m_B}, \tag{27.27}$$

we get the final expression for the kinetic energy as

$$\boxed{\text{KE } = \frac{1}{2}\mu\dot{\vec{x}}^2 + \frac{1}{2}M\dot{\vec{X}}^2.} \tag{27.28}$$

The quantity μ is known as the reduced mass of the two particle system.

The total kinetic energy is a sum of two parts:

(a) a part $\frac{1}{2}\mu\vec{v}^2$ associated with the relative motion;

(b) a second part $\frac{1}{2}M\vec{V}^2$ associated with motion of the centre of mass of the system.

§6 Angular momentum

The total angular momentum of the system is

$$\vec{L} = m_A\vec{x}_A \times \vec{v}_A + m_B\vec{x}_B \times \vec{v}_B. \tag{27.29}$$

To express the above expression terms of relative and centre of mass variables we proceed as in the case of kinetic energy. Using Eq.(27.7). first everything is rewritten in terms of $\vec{X}_A, \vec{X}_B$ and then Eq.(27.19) is used to get the final from in terms of $\vec{X}$ and $\vec{X}$.

$$\begin{aligned}\vec{L} &= m_A(\vec{X}_A + \vec{X}) \times (\vec{v}_A + \vec{V}) + m_B(\vec{X}_B + \vec{X}) \times (\vec{v}_B + \vec{V}) \\ &= m_A\Big(-\frac{m_B}{M}\vec{x} + \vec{X}\Big) \times \Big(-\frac{m_B}{M}\vec{v} + \vec{V}\Big) + m_B\Big(\frac{m_A}{M}\vec{x} + \vec{X}\Big) \times \Big(\frac{m_A}{M}\vec{v} + \vec{V}\Big)\end{aligned} \tag{27.30}$$

Expanding and simplifying, some terms cancel and collecting similar terms we get

$$\begin{aligned}\vec{L} &= \Big(\frac{m_A m_B^2}{M^2} + \frac{m_B m_A^2}{M^2}\Big)(\vec{x} \times \vec{v}) + (m_A + m_B)(\vec{X} \times \vec{V}) \\ &= (m_A m_B)\frac{m_A + m_B}{M^2}(\vec{x} \times \vec{v}) + (m_A + m_B)(\vec{X} \times \vec{V}) \\ &= \frac{m_A m_B}{m_A + m_B}(\vec{x} \times \vec{v}) + (m_A + m_B)(\vec{X} \times \vec{V}).\end{aligned} \tag{27.31}$$

Thus we get the final form for the angular momentum as

$$\boxed{\vec{L} = \mu\vec{x} \times \vec{v} + M\vec{X} \times \vec{V}}. \tag{27.32}$$

Thus the total angular momentum is a sum of two parts:

(a) angular momentum $\mu\vec{x} \times \vec{v}$ associated with relative motion;

(b) angular momentum $M\vec{X} \times \vec{V}$ associated with motion of the centre of mass of the system.

In the centre of mass frame, defined as the frame in whcih the centre of mass is at rest we have the following expressions.

$$(\text{KE})\Big|_{\text{cm}} = \frac{1}{2}\mu\vec{v}^2, \qquad (\vec{L})\Big|_{\text{cm}} = \mu\vec{x} \times \vec{v}. \tag{27.33}$$

Lesson 28

Qualitative Discussion of Central Force Motion

Oct 17, 2012

§1 Introduction

In this lecture we begin a discussion of derivation of planetary orbits. As explained in the last lecture the two body problem of motion the Sun and the Earth can be reduced to free motion of the centre of mass, and motion of a single particle of reduced mass and moving in potential $-k/r$,where $k = GmM$.

Viewed from the centre of mass frame both the Sun and the Earth go round the centre of mass in elliptic orbits.

The solution to the equivalent one body problem is also the complete answer for the motion of the Earth, if the Sun motion is ngelected by assuming it to be infinitely heavy.

The equation of the orbit will be derived by making use of conservation laws for energy and angular momentum.

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§2 Conservation laws

We set up the problem of motion of one body of mass μ moving in potential $V(r) = -k/r$. Let $\vec{r}$ denote the position vector of the Earth w.r.t. the Sun. As already explained the motion takes place in a plane and we will use plane polar coordinates r, θ

The energy and angular momentum are constants of motion and we have

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 + V(r) = \text{constant}, \quad (28.1)$$

$$L = \mu r^2\dot{\theta}^2 = \text{constant}. \quad (28.2)$$

Solving for $\dot{\theta}$ from (28.2), and substituting in energy expression we get

$$\dot{\theta} = \frac{L}{\mu r^2} \Rightarrow \dot{\theta}^2 = \frac{L^2}{m^2 r^4} \tag{28.3}$$

$$\therefore E = \frac{1}{2}\mu \dot{r}^2 + \frac{L^2}{2\mu r^2} + V(r) \tag{28.4}$$

Substituting $V(r) = -k/r$ gives

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{L^2}{2\mu r^2} - frackr. \tag{28.5}$$

§3 A qualitative discussion of orbits

Before starting to solve the equations and before deriving equation of orbits we give a qualitative discussion of nature of orbits.

The expressions Eq.(28.4)-(28.5) for energy are similar to the expression $\frac{m}{\dot{x}}^2 + V(x)$ for energy in one dimension, except that the expression $\frac{L^2}{2\mu r^2} + V(r)$ plays the role of potential. We will call this expression *effective potential*:

$$V_{\text{eff}} = V(r) + \frac{L^2}{2\mu r^2}. \tag{28.6}$$

Notice that
the first term in Eq.(28.5) is always positive, we must have

$$E \geq V_{\text{eff}}(r). \tag{28.7}$$

Therefore, the orbit will confined to where the effective potential is less than the total energy E of the system.

A plot of the effective potential for gravitational potential can be drawn and is shown below. For the following discussion we assume $L \neq 0$. As $r \to 0$ the effective potential becomes very large and tends to infinity. For very large values of r the term $L^2/2mr^2$ is negligible compared to $-k/r$ and the effective potential is negative. The effective potential becomes zero at $r = \frac{L^2}{2k\mu}$ and has a minimum at r_0 where r_0 is given by

$$\frac{k}{r^2} = \frac{L^2}{\mu r^3} \Rightarrow r = \frac{L^2}{k\mu} \tag{28.8}$$

and the value of effective potential energy at this r is easily found to be

$$V_0 = -\frac{1}{2}\frac{k}{r}. \tag{28.9}$$

[Q/T]:= Do you recognise Eq.(28.8)? Have you seen this equation earlier? NO?

[R/T]:= OK, I will give a hint. Use the fact that the angular momentum is $L = \mu v r$.

[S/T]:= *YES Sir. This is the same as*

$$\textit{Centrifugal force} \;\; \mu^2 v/r = k/r^2$$

that was used earier in discussion of circular orbits in 12th class.

[R/T]:= *We will soon establish the connection with the circular orbits.*

We will now discuss types of possible orbits depending on the value of the energy E.

§3.1 $E = V_0$,Circular Orbits

When $E = V_0$, the particle can have only one value of r which must be equal to r_0. Also note that for this value of radius corresponds to a minimum, $\frac{d}{dr}V_{\text{eff}} = 0$. Hence the effective radial force is zero and the particle remains at the same radial distance. For this case $E = V_{\text{eff}}(r_0)$ and therefore $\dot{r} = 0$, which tells that the value of r does not change with time. Also Eq.(28.2) implies that the angular velocity is $L/\mu r^2$ is also constant. Thus the particle moves in a circular path with a constant angular velocity.

These results you must have learnt and derived using Newton's laws for circular orbits.

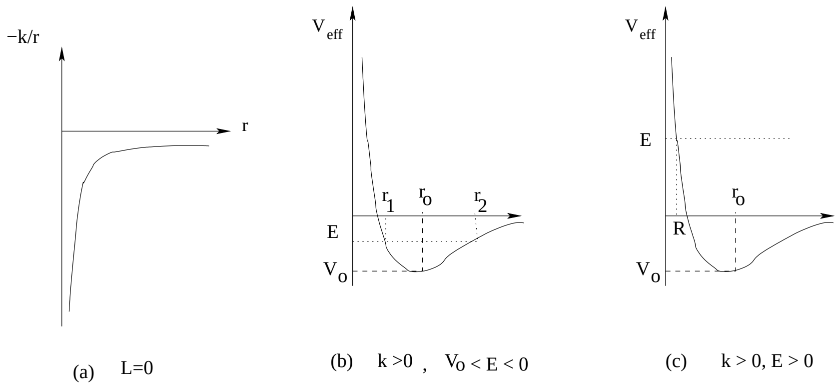

Fig. **28.** Effective potential and allowed ranges of r

§3.2 $V_0 < E < 0$, Bounded motion

When $E < 0$ but greater than the minimum value, the are two values of r for which $E = V(r)$, marked as r_1, r_2 in the figure. Outside this range $r_1 < r < r_2$, the energy is less than the effective potential and therefore the particle cannot go outside the circles $r = r_1$ and $r = r_2$ and the orbits are bounded by these two circles.

Note that at $r =_1, r_2$ the value of the radial velocity $\dot{r}$ is zero and the particle's radial motion is reversed

[Q/S]:= If at $r = r_1$, the radial velocity is zero why does it turn back? Why does it not remain there?

[R/T]:= *I will give a familiar example. When a pendulum swings and reaches the extreme point where its velocity becomes zero, why does it not stay there?*

You send ball rolling up a slope, it goes up a certain distance, stops and then rolls back Why does it not stay where it stops?

[R/S]:= *For the pendulum the force is nonzero at the extreme points.*

[R/T]:= *Nonzero force means nonzero acceleration, $\ddot{x} \neq 0$. Therefore, the velocity $\dot{x}$ cannot remain zero all the time. Similarly, in this case you can show that $\ddot{r}$ is nonzero and has correct sign .*

[Q/S]:= Sir, I do not understand, why you mean by saying correct sign?

[A/T]=: *At the extreme points it can move only in one direction. So you must check at that point the sign of the force, or of the acceleration, which is the same as the sign of the second derivative has value.*

[R/S]:= *So we have to compute the second derivative ?!*

[A/T]=: *YES. But no calculation is actually needed for this. The sign is same as the sign of the slope of the (effective) potential energy at that point.*

Just imagine a particle held at that point and released from rest, which way will it roll down? That is direction of the second derivative.

We will show that for this range of energies, the orbits are elliptic.

§3.3 $E > 0$, Unbounded motion

For $E > 0$ there is one point R such that $E > V_{\text{eff}}$ for all $r > R$. In this case the motion takes place between R and infinity and the orbits are unbounded.

Imagine an electron coming in and approaching a positively charged ion, not a head on collision, it will reach minimum distance R from the atom. At $r = R$ its radial velocity becomes zero, but not the angular velocity, after that it keeps moving away from the ion and then go to infinity.

[Q/T]:= What happens when $L = 0$? Figure it out yourself from the corresponding figure.

Finally it should be noted that for $k < 0$, only unbounded motion is possible. This is the case, for example for Rutherford scattering of α particles from atomic nucleus.

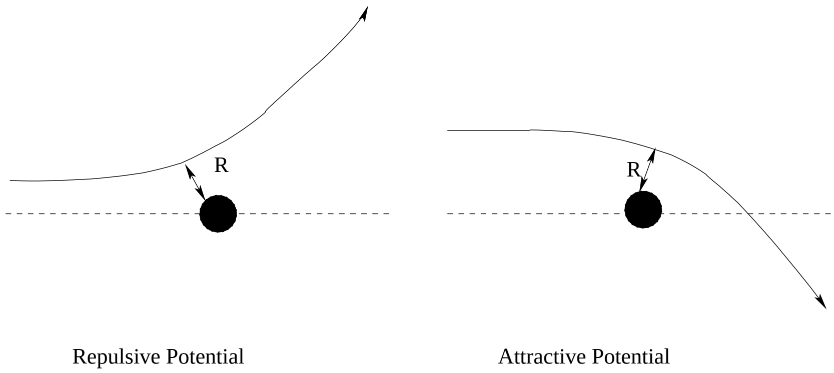

Fig. **29.** Unbounded Orbits in $-k/r$ potential

Lesson 29

Kepler Orbits

Oct 22, 2012

§1 Introduction

We have seen that the EOM for Sun-Earth system can be solved by replacing the two body problem with an equivalent one body problem. For the one body problem the mass should be taken as the reduced mass. Since the Sun's mass is much larger than the mass of the Earth, the reduced mass μ is close to the mass of the Earth to a very good approximation.

We begin with a proof of conservation of angular momentum and its consequences. The conservation of angular momentum implies that (i) the orbit lies in a plane, (ii) areal velocity of the Earth remains constant (Kepler's second law).

Instead of setting up EOM using Newton's laws, the equation of the orbit of the orbit will be derived using the conservation laws. We will, therefore, start with the expressions for the two constants of motion the energy E and angular momentum L and derive a first order differential equation. Integrating the differential equation will give the orbit.

The solution of the equation of the orbit will then be used to prove the third law of Kepler.

§2 Angular momentum conservation

The momentum of a particle $m\vec{x}\times\vec{v}$, depends on the choice of the coordinate system. The obvious choice for the origin of the coordinate is the Sun. The rate of change of angular momentum of a body is given by the torque of forces acting on the body.

$$\frac{d\vec{L}}{dt} = \vec{\tau}. \tag{29.1}$$

Here torque is given by $\vec{\tau} = \vec{x} \times \vec{F}$, where $\vec{x}$ is the position vector of the body and $\vec{F}$ is the net forces acting on it. In case of the Earth's motion the Sun, both these vectors lie along the line joining the Earth and the Sun, Hence their cross product is zero implying $\vec{\tau} = 0$. Thus the angular momentum is a constant of motion.

The angular momentum is a vector quantity, the fact that it is a constant implies that its direction does not change and the magnitude remains constant. We take choose direction to the X_3 axis.The direction of the angular momentum vector $m\vec{x} \times \vec{v}$ is perpendicular to both position vector and velocity. Hence the position vector and velocity vector must always lie in the $X_1 - X_2$ plane.*This gives our first result that the orbit of the Earth lies in a plane.*

We will now prove that one of the consequence of the magnitude of $\vec{L}$ remaining constant is that the Earth sweeps equal areas in equal times at all locations in its orbit.

[Q/T]:= What is the area of a triangle?

[A/T]:= The area of a triangle is given by $\frac{1}{2}ab\sin\theta$, where a, b are lengths of two sides and θ is the angle between the two sides.

[Q/T]:= Correct. Now tell me this answer in vector notation. I give three vectors $\vec{a}, \vec{b}, \vec{c}$ represent a triangle.

[R/S]:= The area of the triangle will be modulus of $\frac{1}{2}\vec{a} \times \vec{b}$.

[Q/T]:= OK. We proceed.

If the position vectors of the Earth at time t is $\vec{x}$ and at time $t + \Delta t$ is $\vec{x} + \vec{\Delta x}$, the area swept in time Δt is then given by

$$\begin{aligned} \text{area swept in time } \Delta t &= \frac{1}{2}|\vec{x} \times (\vec{x} + \vec{\Delta x})| \\ &= \frac{1}{2}|\vec{x} \times (\vec{\Delta x})| \qquad (\because \vec{x} \times \vec{x} = 0). \end{aligned} \tag{29.2}$$

The areal velocity is obtained by dividing the above answer by Δt and taking the limit $\Delta \to 0$, giving

$$\begin{aligned} \text{areal velocity} &= \frac{1}{2}|\vec{x} \times \Big(\frac{\vec{\Delta x}}{\Delta t}\Big)| \\ &= \frac{1}{2}|\vec{x} \times \vec{v}| \\ &= \frac{1}{2m}|\vec{L}|. \end{aligned} \tag{29.3}$$

Thus constancy of the magnitude of angular momentum implies that the areal velocity remains constant, the Kepler's second law.

§3 Equation of the orbit

The motion of the Earth is restricted to a plane, which have chosen to be the $X_1 - X_2$ plane. We will work in plane polar coordinates r, θ as shown in the figure. The energy and angular momentum the are plane polar coordinates are given by

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 - \frac{k}{r}, \tag{29.4}$$

$$L = \mu r^2\dot{\theta}. \tag{29.5}$$

Solving Eq.(29.5) for $\dot{\theta}$ gives

$$\dot{\theta} = \frac{L}{\mu r^2} \Rightarrow \dot{\theta}^2 = \frac{L^2}{\mu r^4}, \tag{29.6}$$

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{k}{r}. \tag{29.7}$$

We want to find the equation of the orbit, which means we must determine r as function of θ. Therefore we regard r as a function $r(\theta)$, and we express $\dot{r}$ in terms of $\frac{dr}{d\theta}$ and proceed as follows.

$$\begin{aligned} \frac{dr}{dt} &= \frac{dr}{d\theta}\frac{d\theta}{dt} \\ &= \frac{L}{2\mu r^2}\frac{dr}{d\theta}. \end{aligned} \tag{29.8}$$

$$\therefore E = \frac{1}{2}\mu\Big(\frac{dr}{d\theta}\Big)^2\Big(\frac{L^2}{\mu r^2}\Big)^2 + \frac{L^2}{2\mu r^2} - \frac{k}{r}. \tag{29.9}$$

Next change variable from r to $u = 1/r$ and use

$$\frac{dr}{d\theta} = \frac{d}{d\theta}(\frac{1}{u}) = -\frac{1}{u^2}\frac{du}{d\theta} \tag{29.10}$$

to get

$$\begin{aligned} E &= \frac{1}{2}\mu\frac{1}{u^4}\Big(\frac{du}{d\theta}\frac{L^2}{\mu r}\Big)^2 + \frac{L^2}{2\mu r^2} - \frac{k}{r} \\ &= \frac{L^2}{2\mu}\Big(\frac{du}{d\theta}\Big)^2 + \frac{L^2}{2\mu}u^2 - \frac{k}{r} \\ &= \frac{L^2}{2\mu}\Big(\frac{du}{d\theta}\Big)^2 + \frac{L^2}{2\mu}u^2 - ku. \end{aligned} \tag{29.11}$$

Complete the square in the last term to get

$$E = \frac{L^2}{2\mu}\Big(\frac{du}{d\theta}\Big)^2 + \frac{L^2}{2\mu}\Big(u - \frac{k\mu}{L^2}\Big)^2 - \frac{k^2\mu}{2L^2} \tag{29.12}$$

$$\tag{29.13}$$

We will now define a new variable σ by means of equation $u = \big(\frac{k\mu}{L^2}\big)\sigma$.

[Q/T]:= What are $[MLT]$ dimensions of σ?

[R/S]:= Sigma is dimensionless.

[R/T]:= That is correct and that is also the reason for introducing this variable.

The energy expression written in terms of σ takes the form

$$\begin{aligned} E &= \frac{L^2}{2\mu}\Big(\frac{k\mu}{L^2}\Big)^2\Big(\frac{d\sigma}{d\theta}\Big)^2 + \frac{L^2}{2\mu}\Big(u^2 - \frac{k^2\mu\sigma}{L^2}\Big) \\ &= \frac{k^2\mu}{2L^2}\Big(\frac{d\sigma}{d\theta}\Big)^2 + \frac{k^2\mu}{2L^2}\sigma^2 - \frac{k^2\mu}{L^2}\sigma. \qquad (29.14) \\ \text{or, } \frac{2EL}{k^2\mu} &= \Big(\frac{d\sigma}{d\theta}\Big)^2 + \sigma^2 - 2\sigma \\ \therefore \frac{2EL}{k^2\mu} &= \Big(\frac{d\sigma}{d\theta}\Big)^2 + (\sigma-1)^2 - 1. \qquad (29.15) \end{aligned}$$

Introducing $\rho = \sigma - 1$, the differential Eq.(29.15) takes the form

$$\frac{2EL}{k^2\mu} + 1 = \Big(\frac{d\rho}{d\theta}\Big)^2 + \rho^2. \qquad (29.16)$$

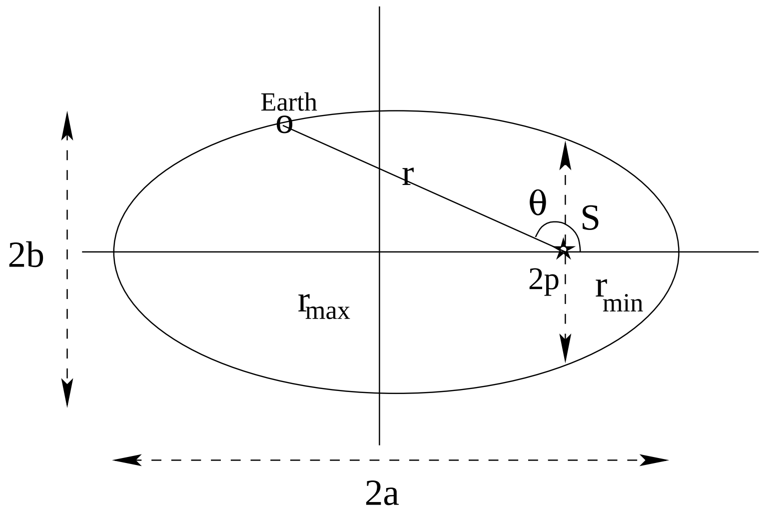

Fig. **30.** Elliptic orbit

The above differential equation can be transformed into a more convenient form for purpose of integration as follows.

$$\begin{aligned} \Big(\frac{d}{d\rho}\theta\Big)^2 &= (e^2 - \rho^2) \qquad (29.17) \\ \text{or } \frac{d\rho}{\sqrt{e^2-\rho^2}} &= d\theta \qquad (29.18) \end{aligned}$$

$$\int \frac{d\rho}{\sqrt{e^2-\rho^2}} = \theta + \text{constt} \qquad (29.19)$$

where

$$e^2 = 1 + \frac{2EL}{k^2\mu} \tag{29.20}$$

Using the standard integration formula

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}(x/a)$$

we get

$$\sin^{-1}\frac{\rho}{e} = \theta + C \tag{29.21}$$
$$\tag{29.22}$$

To get the answer in the standard form, we choose $\theta = 0$ when ρ has the maximum value a. This give $C = \frac{\pi}{2}$. Thus we get

$$\frac{\rho}{e} = \sin(\theta + \pi/2) \tag{29.23}$$
$$\rho = e\cos\theta \tag{29.24}$$

The variable ρ can now be expressed in terms of r by

$$\rho = \sigma - 1 = \frac{L^2}{k\mu}u - 1 \tag{29.25}$$

giving

$$\Big(\frac{L^2}{k\mu}\Big)u = 1 + e\cos\theta \tag{29.26}$$

Changing to $r = 1/u$ and introducing the notation $\ell = \frac{L^2}{k\mu}$, we get the final form for the equation of the orbit

$$\boxed{\frac{\ell}{r} = 1 + e\cos\theta} \tag{29.27}$$

Note that for bounded motion $E < 0$ (why?) and $e < 1$, the above equation then describes an ellipse with the Sun at the focus of the ellipse. For $E = 0$ the orbit is parabolic and for $E > 0$ the motion is not bounded (why?) and the particle moves in a hyperbolic orbit.

Lesson 30

Kinetic Energy Angular Momentum: Many Particle System

Oct 22, 2012

§1 Introduction

In this lecture we shall discuss the kinetic energy and angular momentum of a dynamical many particle system.

Transformation properties of kinetic energy and angular momentum under Galilean transformations is discussed. It is proved that that these dynamical variables can be decomposed into a sum of two parts corresponding to the motion of centre of mass and the motion relative to the centre of mass.

§2 Galillean Transformations

Consider two frames K and K'. Let $\vec{x}$ and $\vec{x}\,'$ be position vectors of a point with respect to K and K'. Let the origin of K' be position $\vec{a}$ w.r.t. the origin of K. Then

$$\vec{x} = \vec{x}\,' + \vec{a}. \tag{30.1}$$

Let $m_{(\alpha)}, \alpha = 1, 2, ...N$ denote the masses, and $\vec{x}_{(\alpha)}, \vec{v}_{(\alpha)}, \vec{p}_{(\alpha)}$ and $\vec{x}\,'_{(\alpha)}, \vec{v}\,'_{(\alpha)}, \vec{p}\,'_{(\alpha)}$ be the position, velocity and momenta of the particle α as seen in the two frames K and $'$. If $\vec{u}$ denotes the velocity of the orign of the frame K' w.r.t. the frame K, then

$$\vec{u} = \frac{d\vec{a}}{dt}. \tag{30.2}$$

and

$$\vec{v}_{(\alpha)} = \vec{v}\,' + \vec{a}, \qquad \vec{p} = \vec{p}\,'_{(\alpha)} + m_{(\alpha)}\vec{u}. \tag{30.3}$$

The total momentum of the system in the frame K is

$$\vec{P} = \sum_{\alpha} \vec{p}'_{(\alpha)} + \sum_{\alpha} +M\vec{u}, \tag{30.4}$$

where M is the total mass of the system and hence

$$\vec{P} = \vec{P}' + M\vec{u}. \tag{30.5}$$

The total kinetic energy is then given by

$$\begin{aligned} T &= \frac{1}{2}\sum_{\alpha} m_\alpha \vec{v}^2_{(\alpha)} \\ &= \frac{1}{2}\sum_{\alpha} m_\alpha (\vec{v}'_{(\alpha)} + \vec{u})^2 \\ &= \frac{1}{2}\sum_{\alpha} m_\alpha (\vec{v}'^2_{(\alpha)} + 2\vec{v}_{(\alpha)} \cdot \vec{u} + \vec{u}^2) \\ &= T + (\sum_{(\alpha)} m_\alpha \vec{v}_{(\alpha)}) \cdot \vec{u} + \frac{1}{2}\sum_{\alpha} m_{(\alpha)} \vec{u}^2) \end{aligned} \tag{30.6}$$

Therefore we get

$$\boxed{T = T' + \vec{P}' \cdot \vec{u}' + \frac{1}{2}M\vec{u}^2} \tag{30.7}$$

Next we obtain the transformation rule for angular momentum.

$$\begin{aligned} \vec{L} &= \sum_{\alpha} \vec{x}_{(\alpha)} \times \vec{p}_{(\alpha)} \\ &= \sum_{\alpha} (\vec{x}'_{(\alpha)} + \vec{a}) \times (\vec{p}'_{(\alpha)} + m_\alpha \vec{u}) \\ &= \sum_{\alpha} \vec{x}'_{(\alpha)} \times \vec{p}'_{(\alpha)} + \sum_{\alpha} m_\alpha \vec{x}'_{(\alpha)} \times \vec{u} + \vec{a} \times \sum_{\alpha} \vec{p}'_{(\alpha)} + (\sum_{\alpha} m_\alpha)\vec{a} \times \vec{u}. \end{aligned} \tag{30.8}$$

Therefore

$$\boxed{\vec{L} = \vec{L}' + M(\vec{X}' \times \vec{u}) + \vec{a} \times \vec{P}' + M\vec{a} \times \vec{u}}. \tag{30.9}$$

where $\vec{X}'$ denotes the position vector of the centre of mass w.r.t. the frame K'.

§3 Special Cases

Case I: K' at rest w.r.t. K

Let the frame K' be at rest w.r.t the frame K but the origins of the two frames do not coincide. Then

$$\vec{a} \neq 0, \qquad \vec{u} = 0. \tag{30.10}$$

Then we have

$$\begin{aligned} T &= T', \qquad (30.11) \\ \vec{L} &= \vec{L}\,' + \vec{a} \times \vec{P}\,'. \qquad (30.12) \end{aligned}$$

Case-II: K' coincides with the centre of mass frame

The centre of mass frame is defined as the frame in which the momentum of the centre of mass is zero, *i.e.* the frame in which the centre of mass is at rest:

$$\vec{P}\,' = 0 \Rightarrow V'_{cm} = 0. \qquad (30.13)$$

The transformation rules become

$$T = T' + \frac{1}{2} M \vec{u}^{\,2} \qquad (30.14)$$

$$\vec{L} = \vec{L}\,' + M \vec{a} \times \vec{u}. \qquad (30.15)$$

Note that the origin of K' may or may not coincide with the position of the centre of mass. Also the axes of the two systems may or may not be parallel.

Case-III : The origin of K' coincides with the c.m.

If in addition to the frame K' moving with the velocity of the centre of mass, let us assume that the origin of K' coincides with the the position of the centre of mass, $\vec{a} = \vec{X}$, then we have

$$\vec{a} = \vec{X} = \frac{\sum_\alpha m_\alpha \vec{x}_{(\alpha)}}{\sum_\alpha m_\alpha}, \qquad \vec{u} = \vec{V} \frac{P}{M}. \qquad (30.16)$$

In this case the transformation rules simplify to

$$\boxed{T = T' + \frac{1}{2} M \vec{V}^{\,2}} \qquad (30.17)$$

$$\boxed{\vec{L} = \vec{L}\,' + M \vec{X} \times \vec{V}}. \qquad (30.18)$$

The above results will be useful in discussion of rigid body dynamics.

Lesson 31

Rigid Body Dynamics-I

Oct 22, 2012

§1 Introduction

The previous lectures today, I know was a bit too fast and I have skipped algebra and calculus steps which you can fill in yourself. I had to do this as we are running short of lectures due to loss of about six classes.

The lectures on the next topic, ”Rigid Body Dynamics” will not be like that. Today I will write a minimum number of equations. I first encountered this subject in my M.Sc. days. In my university we had a system of one final examination at the end of academic year. Typically, we went around looking for syllabus and old question papers, hunt around for books right in the beginning of the session. Whereever possible we tried to make sense of the every subject on our own in advance of class room lectures. Fortunately, Professor P.K. Sharma who taught us Classical Mechanics had been an excellent teacher and we could learn a lot quite well.

In my M.Sc. days We would start working in our own way. There was no guarantee that the syllabus will be covered fully in all the courses. There was always a possibility that the classes will be disturbed due to student strikes etc. Even if the syllabus was fully covered, the question paper would typically be set by an external examiner and questions beyond class room lectures would find their way into the final examination papers. So we had to prepare everything and be prepared for more. Almost every one in the class did this exercise.

As far as the rigid body dynamics is concerned, I remember, I could follow most of the steps of derivations and equations. Still many questions were unanswered.

I did not know why things were done the way they were done and why certain things were not treated in some other way. These questions remained unanswered.

That was the end of my classical mechanics course in 1966. The Classical Mechanics course in IIT Kanpur did not touch upon this subject. So I encountered this subject when I had to teach this subject here in UoH in 1981 to the M.Sc. first semester students. I tried to make sense out of the topic and to have a better overall understanding myself. So that was second learning for me, when you teach something, you have a chance to learn the subject again. At that time I found a nice little book very helpful. I think I have already given you reference of this book in connection with rotations and non inertial frames. This is a book from a Russian publisher known as Mir Publishers Moscow. The book costed about Rs 15/= in those days. A copy must be there in library, in case you do not find it let me know. I will place scanned pages from my personal copy and make it available on moodle course site. The integrated section in the library will not have the book but the general section will definitely have a copy. The reference of the book is

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A. S. Kompaneyets, "A Course of Theoretical Physics" Vol-I,
Mir Publishers,Moscow (1978)}
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I strongly recommend this book for non inertial frames and rigid body dynamics. As far as these topics are concerned, even today I think this has the best treatment as far as the physics part is concerned. Typically in any of the standard text books you may have to read 50 pages to extract the main important points to get an overall picture. The book by Kompaneyets tells this part very clearly in about ten pages. I strongly recommend this book and also the books by Landau and Lifshitz and by Sommerfeld.

§2 Arbitrary displacement and degrees of freedom of a rigid body

We have done all the preparation and covered all the mathematics needed for this topic of rigid body dynamics. Therefore, no new preparation is needed.

Q/T:= So let us begin with what is a rigid body? What is your idea of a rigid body? What type of model would you give for a rigid body?

A/S:= It has a lot of electrons protons/ nuclei. It consist of several particles.

A/S:= Ideal rigid body has an infinite modulus of elasticity.

Q/T:= An ideal rigid body consist of a large number of particles whose distances do not change. If you apply some stress, there is no change in shape or in length or volume of the body. That is what Vidur meant when he said infinite elasticity.

So let us think of a rigid body as consisting of several particles labelled by index α. Let $m_\alpha, \vec{x}_\alpha, \vec{v}_\alpha, \vec{p}_\alpha$ denote the corresponding mass, position, velocity and momentum etc. Typically, a rigid body, like this duster, will have 10^{23} particles and distances between them are fixed. Now we want to know what kind of motion is involved when we take a rigid body from one position to another one in an arbitrary manner. I quote the answer contained in the following theorem as given in Whittaker's book on Analytical Dynamics.

Theorem 4 (Chasels' Theorem). *The most general displacement of rigid body can be obtained by first translating the body and then rotating it about a line.*

So take this block of wood carry it around, move it the way you like. The final position can be always obtained by performing two operations on the body. Starting from any initial position we can take it to an arbitrary final position by first translating it and then by rotating it about an axis by some angle.

The translation is given by three parameters. Typically we can select a point on the body and we note down its initial and final positions. The difference between the two position vectors, $\vec{x}_i$ and $\vec{x}_f$ will give the required translation. In fact, in many cases this point will be taken to be the position of the centre of mass of the rigid body. Three numbers will be needed to specify the translation.

A rotation, as you already know, is specified by an axis of rotation and an angle of rotation. The axis of rotation is specified by a unit vector $\hat{n}$ which has three parameters n_1, n_2, n_3 satisfying one relation $n_1^2 + n_2^2 + n_3^2 = 1$. So the choice of axis requires two independent parameters. These two together with the angle of rotation make the number of independent parameters required to be three.

A rotation can also be, equivalently specified by means of three suitably selected angles. One such set of angles is called **Euler angles.**

We have done the counting of number of degrees of freedom of a rigid body in an earlier lecture and we found it to be six. That is what is being said again today.

§3 State specification

Q/T:= Now the next question is how many variables are needed to specify the state of a rigid body? The state is a different concept and specifying state is not the same thing as specifying the most general displacement. So how many parameters are needed to specify the state? Remember what happens for a point particle. The concept of state is an important one and it will keep coming to you in all theories such a thermodynamics, fluid mechanics, quantum mechanics etc..

Q/T:= So if you have a particle its position is given by three coordinates. What about its state?

A/S:= There components of potion of a point particle.

R/T:= No! Any other answer? We have discussed the concept of the state in detail.

A/S:= The state of a point particle is specified by three position coordinates and three components of its velocity

Q/T:= So what about the state of a rigid body?

R/T:= We require six coordinates and six components of velocities. What are these six components of velocities? These are three components of velocity of a chosen point on the rigid body and three components of angular velocity. The latter three velocities tell us how fast the angle of rotation is changing and how fast the axis of rotation itself is changing the direction.

If I know these six quantities, assuming the laws of motion, and the forces and the torques, we can compute and predict how the body will move and what will be its position and orientation at a later time.

Q/S:= Should we not include the forces and torques in the state specification?

R/T:= No, the forces and the torques are not part of state specification. Only those quantities are included which are properties of the system. For a fixed system, the forces etc could be different in different situations. For example, the same electron will experience different forces in an electric field and in magnetic fields.

R/T:= Given the state alone we cannot predict the state at a later time unless we know the forces. So "Should we not include the forces etc in the state ?"

R/T:= The state at a given time is the complete information about the system at that time and laws should allow us to compute anything about the system at that time. For finding the state at a later time we also need the laws of motion and the interactions, but the laws and the interactions are not part of the specifying state of the system.

A rigid body under one set of forces and torques will move in one way and the same rigid body will move in some other way under a different set of forces and torques. So the forces and torques determine how the body will move the laws allows us to compute and predict this motion. These two concepts are not a part of state specification.

Recall that in particle mechanics, although velocity and acceleration are defined in terms of the *3 components of the displacement of a particle*, we have to switch to the *position vector* which is needed to specify the position of the particle. The equations of motion are actually written in terms of the *position vectors and not in terms of displacements*. Of course knowing the initial and final position vectors, we know the displacement is given as the difference. We also need the three velocities which come from the rate of change of the position vectors.

Turning to the state of a system of a point particle, and to summarise, the state requires knowledge of position vectors and velocities and not the displacements. Given these we can compute all other physical quantities such as kinetic energy, angular momentum of the system.

In a similar fashion an arbitrary displacement of a rigid body is a translation and a rotation and now the question will be what are the corresponding "coordinates" so that the coordinates and their time derivatives will specify the state of a rigid body. Knowing these at a given time we should be able to calculate all dynamical variables such as kinetic energy and angular momentum pf the rigid body.

§4 Space and body axes

In order to make a transition from displacement of a rigid body to the state of rigid body, we have to ask what coordinates are needed to describe the position and the orientation of the rigid body?

The translational state can be described by means of the position vector $\vec{x}_0$ of a selected point P on the rigid body w.r.t. an inertial frame K_i. The orientation is described by attaching a set of axes K_b in the body with the origin at the point P. These axes are fixed in the body and are called *body axes*. When the body moves these axes also move. We can remove all reference to the body because the rotational motion of the body is completely known in terms of the changes in orientation of the body axes. The body axes are related to the axes K_i, fixed in space, by a translation and a rotation about some axes $\hat{n}$ by an angle θ. The frame of reference fixed in the body will be called body frame, K_b and corresponding axes will be called body axes.

Thus we have described a set of six "coordinates" as the position vector $\vec{x}_0$ and the axis of rotation $\hat{n}$ and the angle of rotation θ and their time derivatives. These twelve quantities are sufficient to completely specify the state of a rigid body.

It may be remarked that the axis and the angle of rotation, $\hat{n}, \theta$, needed to describe the orientation of the body axes w.r.t. the space axes can be replaced with three angles. The three angles for initial and for the final states of a rigid body will give its rotational displacement. This rotational displacement for infinitesimal times will define corresponding three angular velocities.

Given the space and body axes, let the position vectors of points in the body with respect to the two sets of axes be denoted by $\vec{x}_\alpha$ and $\vec{X}_\alpha$ respectively. Then

$$\vec{X}_\alpha = \vec{x}_\alpha - \vec{x}_0. \tag{31.1}$$

In the inertial frame, the Newton's laws give the EOM in the form

$$m_\alpha \frac{d^2\vec{x}_\alpha}{dt^2} = \vec{F}_\alpha^{\,\text{tot}}. \tag{31.2}$$

for each particle α. But these equations are useless for us. WHY? We are not interested in each and every particle, a body under consideration may have 10^{23} particles, their positions and velocities are not independent. The total force $\vec{F}\alpha^{\text{tot}}$ must include the force on α^{th} particle must include the forces due to all other particles. All these forces are to be taken into account if we ever wish to solve the equations Eq.(31.2). All this is too complicated and unnecessary.

We know that finally only 12 quantities are independent; there are six coordinates and in all six of velocities and angular velocities. That is all the information that we need to know. So we should somehow isolate the equations involving the independent six coordinates and corresponding (angular) velocities. We do not want anything else; we do not want 10^{23} equations. This is where something that we have already done will be helpful.

It is sufficient if we know where the selected point of the body, the origin of the body axes, will lie at a given time and corresponding velocity. Also we need the orientation and angular velocities. In fact in very many applications the interest lies only in the angular velocities *i.e.* how fast the system is rotating and not in the orientation itself.

We want to write the EOM so that we can concentrate on the quantities listed above. We do not want anything else to complicate our calculation; we do not want 10^{23} equations. So what are these small number of equations. We already have the answer from earlier lectures.

We recall that the equation of motion for the centre of mass, in the inertial frame, becomes

$$M\frac{d\vec{x}_{cm}}{dt} = \vec{F}^{\,\text{ext}}. \tag{31.3}$$

It may be noted that only net external force appears here and the contribution of internal forces, frequently unknown, cancels out. This is a great simplification.

The remaining information, how the body is rotating, is contained in the EOM for the total angular momentum:

$$M\frac{d\vec{L}}{dt} = \vec{\tau}^{\,\text{tot}}. \tag{31.4}$$

where only the contribution of all, external and internal, forces to the torque is to be taken into account in order to set up the above equation.

However for a rigid body there is further simplification. The contribution of the of the internal forces to the torque in the right hand side of Eq.(31.3) cancels out. Thus we can write

$$M\frac{d^2\vec{L}}{dt^2} = \vec{\tau}^{\,\text{ext}}. \tag{31.5}$$

A proof that this can be done for rigid bodies will be given separately in a later lecture. No proofs are being discussed today.

The above six equations, Eq.(31.4)-Eq.(31.5), are sufficient for obtaining all the information we need to know about motion of a rigid body.*It must be remembered that these are not new EOM, because they follow from the Newton's laws. We have combined the EOM for separate particles Eq.(31.1) and have written the result into a simple, nice and useful form.* Eq.(31.5) determines three angular velocities of interest.

Both the equations do not require knowledge of internal forces between different parts of the rigid body. They involve only external forces and their torques and hence they can be set-up easily. These equations contain all the information of interest about the rigid body dynamics.

Now comes the next point. The first equation, Eq.(31.3), involves only the centre of mass position and velocity and we know how to set it up. The second equation Eq.(31.5) involves the angular momentum of the rigid body and we must ask what is the expression for angular momentum of a rigid body. You have already studied a little about the angular momentum of a rigid body in earlier classes. Keep that aside because that does not give complete and most general answer. I am going to give you my answer which is applicable to every case and reduces to the familiar expression you have learnt in special cases. Here it is not possible to start from a special case and to visualise the answers for general situations. If you keep moving on streets of Hyderabad you can never guess how the city looks like from the sky. But you go to the top of a hill, say near the planetarium you can have a bird's eye view of the city. From a satellite appropriately equipped you can draw the road map, but from the roads it is not so easy to imagine the view as seen from a satellite.

At this point I will discuss three cases separately.

§5 Case I : Body free to move under external forces

Here free to move means no point of the body is fixed.

Without a loss of generality, and in order to simplify the discussion we usually take the origin of the body frame at the centre of mass of the rigid body. At this stage the decomposition of the angular momentum into contributions coming from the centre of mass and that coming from motion of the particles in the rigid body relative to the centre of mass helps. This has already been worked out and the answer is given by

$$\vec{L} = M\vec{x}_{\text{cm}} \times \vec{v}_{\text{cm}} + \sum_{\alpha} m_\alpha \vec{y}_\alpha \times \dot{\vec{y}}_\alpha \tag{31.6}$$

Here $\vec{x}_{\text{cm}}, \vec{v}_{\text{cm}}$ represent the position and momentum of the centre of mass (w.r.t. the inertial frame K_i) and $\vec{y}_\alpha \equiv \vec{x}_\alpha - \vec{x}_{\text{cm}}$ is the position of α^{th} particle relative to the centre of mass.

The Newtonian EOM for the total angular momentum neatly separate into two equations. The first EOM refers to the centre of mass

$$\frac{d\vec{L}_{\text{cm}}}{dt} = \vec{\tau}^{\,\text{ext}} \tag{31.7}$$

where

$$\vec{L}_{\text{cm}} = M\vec{x}_{\text{cm}} \times \vec{v}_{\text{cm}}. \tag{31.8}$$

This equation is already contained in the EOM for the position

$$M\frac{d^2\vec{x}_{\text{cm}}}{dt^2} = \vec{F}^{\,\text{ext}}. \tag{31.9}$$

The second equation written in an inertial frame, such as space axes,

$$\frac{d\vec{L}_{\text{cm}}}{dt} = 0, \tag{31.10}$$

gives the conservation of the angular momentum of the *rigid body* relative to the centre of mass.

§5.1 Force Free Motion

In a special case when the net force and net torque are both zero, the resulting equations can be solved and analytic solutions can be obtained. The motion can also be described in terms of an elegant geometrical construction, known as the Poinsot's construction, using conservation laws.

In the case of force free motion the conservation laws are sufficient to describe the motion of the rigid body completely without the need for analytical solution.

§6 Case II: One point of the body is fixed

In this case it is natural to take the fixed point as the common origin of the body and space frames. The EOM in the inertial frame is given by Eq.(31.5).

The total angular momentum of the body is given by

$$\vec{L} = \sum_{\alpha} m_\alpha \vec{x}_\alpha \times \vec{v}_\alpha. \tag{31.11}$$

Here the sum is over all particles. Does this mean that every time I want the value of angular momentum of a rigid body, I must write all positions, get all the velocities and sum over all the particles? In all practical cases all this is sufficient but unnecessary and also too cmplicated to handle. The situation is saved by the fact that the value of the angular momentum can be written in terms of the position and velocity of the centre of mass, the three angular velocities and the moment of inertia tensor. In terms of these variables the components of angular momentum (31.11)take the form

$$L_j = \sum_{j,k=1}^{3} I_{jk}\omega_k, \qquad j = 1, 2, 3. \tag{31.12}$$

where the quantities I_{jk}, the nine components of moment of inertia tensor, are given by

$$I_{jk} = \sum_{\alpha,\beta} m_\alpha \big(x_j^2 \delta_{jk} - x_{\alpha j} x_{\alpha k} \big). \tag{31.13}$$

Here δ_{jk} denotes the Kronecker delta symbol defined by

$$\delta_{jk} = \begin{cases} 1, & \text{if } j = k \\ 0, & \text{if } j \neq k. \end{cases} \tag{31.14}$$

The EOM in the inertial frame takes the form

$$\frac{d}{dt} \sum_{k=1}^{3} I_{jk}\omega_k = \tau_j^{\text{ext}}, \qquad j = 1, 2, 3. \tag{31.15}$$

Although this equation correctly describes the variation of the angular velocities, a closer look shows that it is a difficult task to solve this equation for the angular velocities. The components of position of different particles constituting the rigid body, *i.e.* $x_{j\alpha}$, vary with time and have a complicated implicit dependence on the angular velocities. This complication is handled by going to the body frame. However the body frame being non inertial the Newton's laws do not apply, Therefore, we must start from the EOM Eq.(31.15) written in the inertial frame and perform a transformation to the body frame. The EOM written in terms of body axes take the form

$$\frac{d\vec{L}^b}{dt} + \vec{\omega} \times \vec{L}^b = \vec{\tau}_b^{\,ext}, \tag{31.16}$$

and the components of the angular momentum in the body frame, L_j^b, are given by

$$L_j^b = \sum_{j,k=1}^{3} I_{jk}^b \omega_k, \qquad j = 1, 2, 3. \tag{31.17}$$

Notice that now the moment of inertia tensor components, w.r.t. the body axes, will not vary with time and will be a set of numbers depending only on the geometrical shape of the rigid body. In EOM Eq.(31.17) the unknown angular velocities appear explicitly and have to be solved for a given problem.

§7 Case III : Rotation of a body with two points fixed

In this case the body rotates about the axis passing through the two fixed points. This is the special case that you must have studied in the +2 standard Physics courses. Some well known examples are flywheel, Kater's pendulum. In these examples, there are some surprises in store for you and one such example will be discussed in the next lecture.

Lesson 32

Rigid Body Dynamics-II

Oct 27, 2012

§1 The story so far

I will first summarize the main points of the overview of the rigid body dynamics from the last lecture. After that I will come to the discussion of the dynamics, Euler EOM, moment of inertia and other details.

1. A rigid body can be modelled as a collection of a large number of particles with distance fixed.
2. The number of degrees of freedom of a rigid body is six. This means there are six independent ways in which it can be displaced. Three displacements correspond to translation along the three axes and the remaining three displacements correspond to rotations about the three coordinate axes.
3. At any time, the body at rest is completely described by six *generalized coordinates* and an arbitrary displacement is unworn if the initial and final *generalized coordinates* are specified. I am calling these as generalised coordinates and generalised velocities as they could be something other than position, velocities for example angles and angular velocities.
4. In general, for a body in motion, the state is given by six *generalized coordinates* and six *generalized velocities.* These six velocities are the three components of velocity of a chosen point on the rigid body and three components angular velocity of the body. Notice that the angles and angular velocities are treated as on par with coordinates and velocities as far describing the state is concerned.

5. The EOM are

$$M\frac{d^2\vec{X}_{\rm cm}}{dt^2} = \vec{F}_{\rm tot} \tag{32.1}$$

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{\rm tot} \tag{32.2}$$

$\vec{X}_{\rm cm}$ is the position vector of the centre of mass and $\vec{L}$ is the total angular momentum of the body.

6. If $\vec{x}\alpha$ and $\vec{v}_\alpha$ are the position and velocity of the point α of the rigid body,

$$\vec{L} = \sum_\alpha m_\alpha \vec{x}_\alpha \times \vec{v}_\alpha. \tag{32.3}$$

The position and velocity values here refer to a chosen inertial frame, which will also be called *space frame* of reference. They depend on time in a complicated way. Also the total angular momentum involves a sum over all parts of the body —— some thing too complicated to handle. Both these problems are solved by introducing a body frame with the origin selected at the center of mass.

7. To an observer in the centre of mass frame the body appears at rest all the positions $\vec{X}_\alpha$ are independent of time and all the velocities of all parts of the body, as seen by the observer, in the body frame are zero.

The velocity of different parts as seen by an observer in the inertial frame $\vec{v}_\alpha$ have simple expressions:

$$\vec{v}_\alpha = \vec{v}_{\rm cm} + \vec{\omega} \times \vec{X}_\alpha. \tag{32.4}$$

8. Here $\vec{X}_\alpha$ denotes the position and and $\vec{V}_\alpha$ denotes relative velocity w.r.t. the centre of mass of the body as seen by the observer in the inertial frame and we have

$$\vec{V}_\alpha = \vec{\omega} \times \vec{X}_\alpha \tag{32.5}$$

9. So we write the EOM as seen by an observer in the space fixed frame, and reexpress them in terms of the center of the position $\vec{X}_{\rm cm}$ and ω using and but use the relative position variables $\vec{X}_\alpha$. The total angular momentum then becomes

$$\vec{L}_{\rm tot} = \sum_\alpha m_\alpha \vec{X}_\alpha \times \vec{V}_\alpha. \tag{32.6}$$

This completes the summary of main points of the last.

Now we substitute expression (32.5) for velocities and rewrite everything in terms of the positions $\vec{X}_\alpha$ and the angular velocity ω. The EOM then assumes the final form

$$\boxed{\frac{d\vec{L}_{\rm tot}}{dt} = \frac{d\vec{L}_b}{dt} + \omega \times \vec{L}_b = \vec{\tau}_{\rm tot}}. \tag{32.7}$$

Here $\vec{L}_b$ is the *total* angular momentum of the body expressed in terms of the body variables $\vec{X}_\alpha$ and the velocities $\vec{V}_\alpha$ relative to the centre of mass. We then have

$$\begin{aligned}
\vec{L}_b &= \sum_\alpha m_\alpha \vec{X}_\alpha \times \vec{V}_\alpha && (32.8)\\
&= \sum_\alpha m_\alpha \vec{X}_\alpha \times (\vec{\omega} \times \vec{V}_\alpha) && (32.9)\\
&= \sum_\alpha m_\alpha \Big[|\vec{X}_\alpha|^2 \vec{\omega} - (\vec{X}_\alpha \cdot \vec{\omega})\vec{X}_\alpha\Big] && (32.10)
\end{aligned}$$

Hence the three components $L_k, k = 1, 2, 3$ of the angular momentum are given by

$$L_{bj} = \sum_\alpha m_\alpha \Big(|\vec{X}_\alpha|^2 \omega_j - (\vec{X}_\alpha \cdot \vec{\omega}) X_j\Big) \qquad (32.11)$$

Remember L_{bj} is not the angular momentum of the body as seen by the observer in the body frame; in the body frame the body is at rest and hence the angular momentum is zero. It can be shown that the three components of the angular momentum can be rewritten in the form

$$L_{bj} = \sum_{k=1}^{3} I_{jk} \omega_k. \qquad (32.12)$$

where I_{jk}, the nine components of the moment of inertia tensor I, are given by

$$\boxed{I_{jk} = \sum_\alpha m_\alpha \Big(|\vec{X}_\alpha|^2 \delta_{jk} - X_{\alpha k} X_{\alpha j}\Big)}. \qquad (32.13)$$

Here $\vec{X}_\alpha$ is the position vector of the particle α with the centre of mass at the origin and, in the case of a rigid body, it does not carry any time dependence.

Let us explicitly write the expressions for the components of moment of inertia tensor.

$$\begin{aligned}
I_{11} &= \sum_\alpha m_\alpha (x_{\alpha 2}^2 + x_{\alpha 3}^2), & I_{12} = I_{21} &= -\sum_\alpha m_\alpha x_{\alpha 1} x_{\alpha 2}, && (32.14)\\
I_{22} &= \sum_\alpha m_\alpha (x_{\alpha 3}^2 + x_{\alpha 1}^2), & I_{23} = I_{32} &= -\sum_\alpha m_\alpha x_{\alpha 2} x_{\alpha 3}, && (32.15)\\
I_{33} &= \sum_\alpha m_\alpha (x_{\alpha 1}^2 + x_{\alpha 2}^2) & I_{31} = I_{13} &= -\sum_\alpha m_\alpha x_{\alpha 3} x_{\alpha 1}, && (32.16)
\end{aligned}$$

For a continuous mass distribution, the mass is to be replaced by ρd^3x where $\rho(\vec{x})$ is the density at position $\vec{x}$ density and the sum over all particles should be replaced by an integral over all volume of the rigid body. So for example, we have

$$I_{11} = \iiint (x_{\alpha 2}^2 + x_{\alpha 3}^2) \rho(\vec{x})\, d^3x; \qquad I_{12} = -\iiint (x_{\alpha 1} x_{\alpha 2}) \rho(\vec{x})\, d^3x. \qquad (32.17)$$

.

§1.1 Proof of Eq.(32.26)

We first define Kronecker delta, δ_{jk}, by

$$\delta_{jk} = \begin{cases} 1, & \text{if } j = k \\ 0, & \text{if } j \neq k. \end{cases} \tag{32.18}$$

A basic manipulation involving the Kronecker delta is

$$\sum_{k=1}^{3} \delta_{jk} B_k = B_j \tag{32.19}$$

This equation follows from the fact that only one term in the summation over k is nonzero; this term corresponds to $k = j$. and in that term the Kronecker delta will be δ_{jj} which equals 1.

Let us multiply Eq.(32.19) by A_j and sum over all values of j. This gives

$$\begin{aligned} \sum_{j=1}^{3} A_j \sum_{k=1}^{3} \delta_{jk} B_k &= \sum_{j=1}^{3}\sum_{k=1}^{3} A_j B_k \delta_{jk} = \sum_{j=1}^{3} A_j B_j \qquad \text{(using Eq.(32.19))} \\ &= \vec{A}\cdot\vec{B} \end{aligned} \tag{32.20}$$

Thus we get a useful representation for the scalar product of two vectors $\vec{A}, \vec{B}$

$$\boxed{\vec{A}\cdot\vec{B} = \sum_{j=1}^{3}\sum_{k=1}^{3} \delta_{jk} A_j B_k}. \tag{32.21}$$

We will use the result Eq.(32.21) to write

$$\vec{X}_\alpha \cdot \vec{\omega} = \sum_{k=1}^{3}\sum_{k=1}^{3} X_{\alpha j}\omega_k \delta jk, \tag{32.22}$$

and

$$\omega_j = \sum_{k=1}^{3} \delta_{jk}\omega_k. \tag{32.23}$$

Therefore, substituting Eq.(32.22), (32.23) in (32.10), the total angular momentum expression becomes

$$L_{bj} = \sum_\alpha m_\alpha \Big(|\vec{X}_\alpha|^2 \omega_j - (\vec{X}_\alpha \cdot \vec{\omega}) X_j \Big) \tag{32.24}$$

$$= \sum_\alpha \sum_k m_\alpha \Big(|\vec{X}_\alpha|^2 \delta_{jk}\omega_k - X_{\alpha k}\omega_k X_{\alpha j} \Big)$$

$$= \sum_\alpha \sum_k m_\alpha \Big(|\vec{X}_\alpha|^2 \delta_{jk} - X_{\alpha j} X_{\alpha k} \Big)\omega_k \tag{32.25}$$

which leads to the desired result

$$L_{bj} = \sum_{k=1}^{3} I_{jk}\omega_k. \tag{32.26}$$

§2 Moment of Inertia Tensor

The moment of inertia tensor has been defined with a choice of axes which had the origin at the centre of mass of the body. The knowledge of moment of inertia tensor allows us to compute the kinetic energy and angular momentum of the body rotating about an axis through the center of mass with angular velocity ω.

The value of these components depend only on the shape and the mass distribution of the body and the choice of coordinate axes. The values of the inertia tensor components change when the coordinate axes are changed.

To write expressions for the kinetic energy and angular momentum of rotation about an axis that does not pass through the origin, we need to translate the axes so that the origin is moved to a new position so that the new origin lies on the axis of rotation. This operation on the axis be done without changing the orientation of the axes. The new components of the moment inertia tensor components can then be easily worked out and result is given separately.

If the axes are rotated, the relation between components w.r.t. the old and new axes can be written down in terms of the axis and angle of rotation. These relations are suitable generalisations of the corresponding relations for the components of a vector already discussed.

Example 16: Let us consider a rigid body consisting of two masses M_1 and M_2 located at $(-a, b, 0), (a, b, 0)$ connected by a rod of negligible mass and bent to from a "V" shape with the bend at the origin. Compute all the moment of inertia tensor of the this body.

[R/T]:= One of you should come to the board and work out the inertia tensor.
The answer is as follows

$$I_{11} = M_1 b^2 + M_2 a^2; I_{22} = M_1 a^2 + M_2 b^2; I_{33} = 0 \tag{32.27}$$

$$I_{12} = I_{21} = (M_1 - M_2)ab; \quad I_{13} = I_{23} = 0. \tag{32.28}$$

It may be recalled that the moment of inertia tensor is a symmetric tensor and $I_{jk} = I_{kj}$.

The nine components of the inertia tensor can be conveniently assembled into a 3×3 matrix $\underline{I}$

$$\underline{I} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}. \tag{32.29}$$

If we write the angular momentum and angular velocity as columns:

$$\underline{\mathsf{L}} = \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix}, \qquad \underline{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \tag{32.30}$$

Then the angular momentum components are given by a matrix multiplication

$$\underline{\mathsf{L}} = \underline{\mathsf{I}}\underline{\omega}. \tag{32.31}$$

[R/T]:= Is angular momentum always parallel to angular velocity? Check this statement by computing the angular momentum of the body in the previous example in three cases of rotation with angular velocity ω with axis of rotation chosen as:

(*a*) X_1; (*b*) X_2; (*c*) X_3.

Example 17: You must have by now done experiments involving fly wheel and Kater's pendulum in your mechanics laboratory. A fly wheel rotating about its axis in your laboratory experiment, or Kater's pendulum oscillating in a vertical plane are examples of a rigid body rotating about a fixed axis. Let us see the use of moment of inertia tensor to compute the moment of inertia and the kinetic energy for these examples.

Let $\hat{n}$ be unit vector parallel to the axis of rotation ON, see Fig.31 below. Let P be a point on the body and $\vec{x}$ be its position vector. Let ρ be the distance of the point P from the axis of rotation. Then

$$\vec{x}^2 - (\hat{n}\cdot\vec{x})^2 = OP^2 - OQ^2 = PQ^2 = \rho^2. \tag{32.32}$$

It is clear that the above expression holds for every part of the body and that

$$\sum_{jk=1}^{3} I_{jk} n_j n_k = \sum_{\alpha} m_\alpha \Big(\vec{x}_\alpha^2 - (\hat{n}\cdot\vec{x}_\alpha)^2\Big) = \sum_{\alpha} m_\alpha \rho_\alpha^2, \tag{32.33}$$

coincides with the standard expression for moment of inertia about the axis; and the kinetic energy will be equal to $\frac{1}{2}I\omega^2$. You must have already used this expression for your experiments and applied to various problems in the 11th and 12th classes.

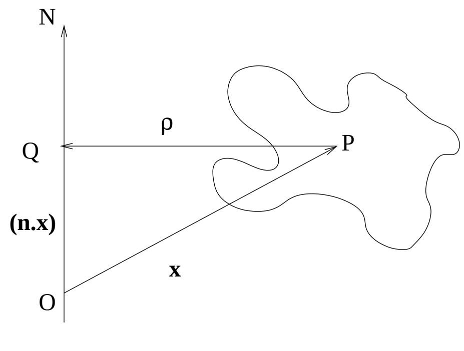

Fig. **31.**

Q/S:= *We have derived expression for moment of inertia and kinetic energy in the body frame assuming that the origin of the axes to lie on the body. So why are those expressions correct for this case, when the origin does not lie on the body the frame of reference is not the one in which the body is at rest?*

R/T:= I do not have an immediate answer and will discuss this issue next time.
We have to stop here as the time for our class is over.

Lesson 33

Examples : Moment of Inertia

Oct 27, 2012

Let me first answer the question which Sachin asked last time. When we have a rigid

body is constrained to rotate about a fixed axis, such as a flywheel or a Kater's pendulum, it is not necessary to introduce to body axes. *The origin must be chosen at a point on the axis of rotation.* The axis of rotation can be taken as one of the coordinate axis; if we do make this choice, every point of the body moves on a circle having its centre on the axis of rotation.

Let us assume that the axis of rotation is chosen to be X_1 axis, the angular velocity is given by

$$\underline{\omega} = \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} \tag{33.1}$$

and therefore the components of the angular momentum

$$\underline{\mathsf{L}} = \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \underline{\mathsf{I}} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \cdot \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} \tag{33.2}$$

are given by $L_1 = I_{11}\omega$ and I_{11} is

$$I_{11} = \sum_{\alpha} m_\alpha (x^2_{(\alpha)2} + x^2_{(\alpha)3}) \tag{33.3}$$

The suffix (α) in brackets runs over different particles. If we use ρ_α to denote the distance of point α of the rigid body, then

$$\rho^2_{(\alpha)} = (x^2_{(\alpha)2} + x^2_{(\alpha)3})$$

and (11) component of the inertia tensor I_{11} is seen to coincide with the standard expression, $\sum m_\alpha \rho^2_{(\alpha)} \equiv I$, known to you from your earlier classes and the first component of angular momentum has the familiar form $I\omega$. When a body is constrained to rotate about a fixed axis, the components of the total torque about other axes will be zero and the corresponding components of the angular momentum will remain constants of motion.

Example 18: Compute moment of inerta tensor of two particles A, B system of masses m_A and m_B connected by a light V-shaped rod as shown in Fig.32.

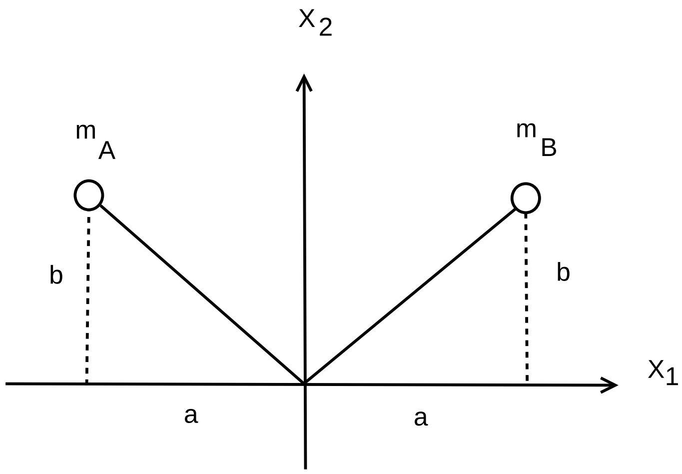

Fig. **32.** Example of MI Calculation

The coordinates of the two masses are

$$\vec{x}_A = (-a, b, 0), \quad \vec{x}_B = (a, b, 0) \tag{33.4}$$

Different components of the moment of inertia tensor are given by

$$\begin{aligned}
I_{11} &= \sum_{\alpha=A,B} m_\alpha (|\vec{x}_{(\alpha)}|^2 - x^2_{(\alpha)1}) && (33.5)\\
&= \sum_{\alpha=A,B} m_\alpha (x^2_{(\alpha)2} + x^2_{(\alpha)3}) \qquad \because |\vec{x}_{(\alpha)}|^2 = x^2_{(\alpha)1} + x^2_{(\alpha)2} + x^2_{(\alpha)3} && (33.6)\\
&= m_A(a^2 + b^2 - a^2) + m_B(a^2 + b^2 - a^2) && (33.7)\\
&= (m_A + m_B) b^2 && (33.8)
\end{aligned}$$

and

$$\begin{aligned}
I_{22} &= \sum_{\alpha=A,B} m_\alpha (|\vec{x}_{(\alpha)}|^2 - x^2_{(\alpha)2}) && (33.9)\\
&= \sum_{\alpha=A,B} m_\alpha (x^2_{(\alpha)1} + x^2_{(\alpha)3}) \qquad \because |\vec{x}_{(\alpha)}|^2 = x^2_{(\alpha)1} + x^2_{(\alpha)2} + x^2_{(\alpha)3} && (33.10)\\
&= m_A(a^2 + b^2 - b^2) + m_B(a^2 + b^2 - b^2) && (33.11)\\
&= (m_A + m_B) a^2, && (33.12)
\end{aligned}$$

$$
\begin{aligned}
I_{33} &= \sum_{\alpha=A,B} m_\alpha(|\vec{x}_{(\alpha)}|^2 - x^2_{(\alpha)3}) && (33.13)\\
&= \sum_{\alpha=A,B} m_\alpha(x^2_{(\alpha)1} + x^2_{(\alpha)2}) \qquad \because |\vec{x}_{(\alpha)}|^2 = x^2_{(\alpha)1} + x^2_{(\alpha)2} + x^2_{(\alpha)3} && (33.14)\\
&= m_A(a^2+b^2) + m_B(a^2+b^2) && (33.15)\\
&= (m_A+m_B)(a^2+b^2). && (33.16)
\end{aligned}
$$

Also

$$
I_{12} = -\sum_{\alpha=A,B} m_\alpha x_{(\alpha)1}x_{(\alpha)2} = -(-m_Aab + m_Bab) = (m_A - m_B)ab, \qquad I_{21} = I_{12}. \quad (33.17)
$$

It is easily seen that $I_{13} = I_{31} = I_{32} = I_{23} = 0$ because $x_{(\alpha)3} = 0$.

Example 19: Show that the moment of inertia of a uniform rod of lenght L, mass M, about an axis passing through its centre of mass and perependicular to its length is $\frac{1}{12}ML^2$.

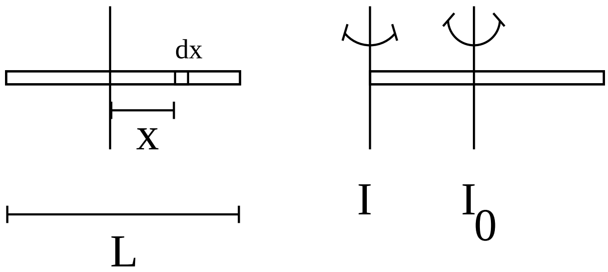

Fig. **33.**

Let the mass per unit length of the rod be λ = Mass/Length, then $M = \lambda L$. We divide the rod into several pieces of small widths. Consider one such piece at position x and width dx. Its mass $= \lambda dx$ and gives a $\lambda dx x^2$ to the moment of inertia. Hence the moment of inertia of the rod is obtained by summing over all pieces. Thus

$$
\begin{aligned}
I_0 &= \int_{-L/2}^{L/2} x^2 \lambda dx\\
&= \lambda \int_{-L/2}^{L/2} x^2 dx\\
&= \lambda \left(\frac{x^3}{3}\right)\Big|_{-L/2}^{L/2}\\
&= \lambda \frac{L^3}{12}\\
&= \frac{1}{12}ML^2. && (33.18)
\end{aligned}
$$

Let I be moment of inertia about an axis to the axis passing through the centre of mass and at a distance d, then

$$I = I_0 + Md^2 \tag{33.19}$$

Thus for an axis perpendicular to the rod and passing through an end point $d = L/2$ and therefore

$$I = \frac{1}{12}ML^2 + \frac{1}{4}ML^2 = \frac{1}{3}ML^2 \tag{33.20}$$

Lesson 34

Angular Momentum : Examples

Oct 29, 2012 In this lecture we will now discuss some examples of applications of the formalism that has been developed.

§1 Rotation about a fixed axis

In this section we will talk about familiar case of a rotation of a rigid body about a fixed axis.

As before we imagine that a rigid body consists of several "small" bodies of mass m_α located at points $\vec{x}^{(\alpha)}$.

When the angular velocity of the body does not change direction, the kientic energy is given by

$$\text{K.E.} = \frac{1}{2} I\omega^2, \tag{34.1}$$

where I is is given by the moment of inertia for this case

$$I = \hat{n} \cdot \underline{I} \cdot \hat{n} = \sum_{jk} n_j I_{jk} n_k \tag{34.2}$$

in terms of the moment of inertia tensor

$$I_{jk} = \sum_\alpha m_\alpha (|\vec{x}^{(\alpha)}|^2 \delta_{jk} - x_j^{(\alpha)} x_k^{(\alpha)}) \tag{34.3}$$

and $\hat{n}$ is unit a vector parallel to the angular velocity. Sachin's queston last time amounts to asking what is the coordinate system that is now being used? Is the origin located at the centre of mass? Several comments will clarify the situation.

First, remember that redrawing some vectors such as torque, angular velocity, parallel to themselves changes physics. We cannot say that the vectors are the same and hence nothing should change. For example magnitude the angular velocity of a moving body and the torque of a force depend on the choice of origin, where as the same is not the case for forces velocity or acceleration. So one has to be careful.

Coming back to the kinetic energy, the expression Eq.(34.1) remains correct as long as the origin lies on the axis of rotation. The expression Eq.(34.2) coincide with the usual expression

$$I = \sum_\alpha m_\alpha R_\alpha^2. \tag{34.4}$$

that you have seen so far. Here R_α is the distance of mass α from the axis of rotation. We will show the equivalnce of the two expressions (34.2) and (34.4) shortly.

Let us then consider a rigid body rotating about a fixed axis. Examples are Kater's pendulum and flywheel, among experiments that you have done in the mechanics laboratory. The axis of rotation may not even pass through the rigid body. It may be a massive body connected to a shaft by means of light rod(s). For example, in figure below. ON is a shaft connected with a heavy rigid body by a light rods AB and DE. Assuming that the shaft rotates with angular velocity ω, we will now compute the kinetic energy of the rigid body and show that the result is same as that given by (34.1)- (34.3).

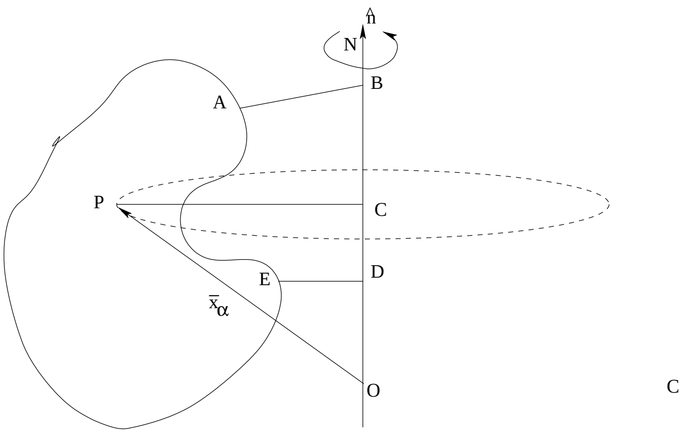

Fig. **34.** Rotation about a fixed axis

Note that each point of the rigid body has the same angular velocity ω and each point moves in a circle. The radius of the circle R_α is the distance of mass α from the axis rotation. The speed of m_α is $R_\alpha \omega$ and therefore the kinetic energy Eq.(34.4) is given by

$$K.E. = \sum_\alpha m_\alpha R_\alpha \omega^2 = I\omega^2, \tag{34.5}$$

where I is given by (34.4).

Let us now consider the expression in Eq.(34.3) for the moment of inertia. Refering to the figure above we see that

$$\vec{x}^{(\alpha)2} = OP^2; \quad \hat{n} \cdot \vec{x}^{(\alpha)} = OP\cos\theta = OC. \tag{34.6}$$

Therefore

$$|\vec{x}^{(\alpha)}|^2 - (\hat{n} \cdot \vec{x}^{(\alpha)})^2 = OP^2 - OC^2 = CP^2 = R_\alpha^2. \quad (34.7)$$

It therefore follows that the expressions (34.3) coincides with (34.4).

§2 Two theorems

Theorem **5** (Parallel axes theorem)**.** *Let I_0 be the moment of inertia of a rigid body about an axis passing through the centre of mass and I be its moment of inertia about a parallel axis at a distance d from the first axis. Then*

$$I = I_0 + Md^2$$

where M is the total mass of the rigid body.

Theorem **6** (Perpendicular axes theorem)**.** *Let the rigid body be in shape of a lamina. Choose the coordinate axes so that body lies in the $X - Y$ plane. If I_x, I_y, I_z denote moment of inertia for rotation about the three axes, then*

$$I_z = I_x + I_y$$

§3 Examples

Example 20: Angular momentum in not always parallel to angular velocity. Compute the angular momentum and kinetic energy of the two masses connected by a V shaped rod of Example 1 in the previous lecture, rotating about different axes with constant angular velocity ω. We will see that the angular momentum is not always parallel to the angular velocity?

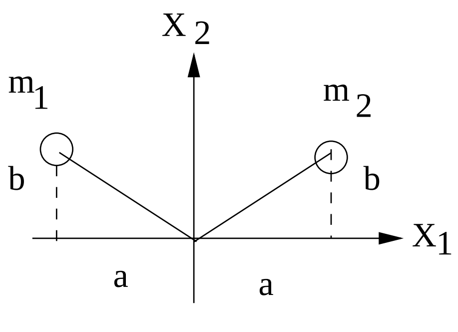

Fig. **35.** Ang velocular Momentum

In this case the moment of inertia tensor was found to be

$$I_{11} = m_1 b^2 + m_2 a^2; \quad I_{22} = m_1 a^2 + m_2 b^2; I_{33} = (m_1 + m_2)(a^2 + b^2). \quad (34.8)$$

Also

$$I_{12} = I_{21} = (m_2 - m_1)ab; \quad I_{23} = I_{32} = I_{13} = I_{31} = 0. \tag{34.9}$$

We take $m_1 = 2$kg, $m_2 = 3$kg, $a = 1$ and $b = 2$m and compute the moment of inertia tensor.

$$\begin{aligned} I_{11} &= m_1 b^2 + m_2 a^2 = 11; I_{22} = m_1 a^2 + m_2 b^2 = 2 + 12 = 14; &(34.10)\\ I_{33} &= (m_1 + m_2)(a^2 + b^2) = 25; I_{12} = -2. &(34.11)\end{aligned}$$

Therefore, the moment of inertia tensor is seen to

$$I = \begin{pmatrix} 11 & -2 & 0 \\ -2 & 14 & 0 \\ 0 & 0 & 25 \end{pmatrix} \tag{34.12}$$

Case I : Angular velocity along the z axis If the system rotates about the z- axis, then the components of angular velocity are given by

$$\underline{\omega} = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \tag{34.13}$$

and the angular momentum is

$$\underline{L} = \underline{I} \cdot \underline{\omega} = \begin{pmatrix} 11 & -2 & 0 \\ -2 & 14 & 0 \\ 0 & 0 & 25 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 25\omega \end{pmatrix} \tag{34.14}$$

Thus we see that the angular momentum is along the z- axis and is therefore parallel to the angular veolcity.

Case II : Angular velocity along the x axis In this case $\vec{\omega} = (\omega, 0, 0)$ and the angular momentum is given by

$$\underline{L} = \underline{I} \cdot \underline{\omega} = \begin{pmatrix} 11 & -2 & 0 \\ -2 & 14 & 0 \\ 0 & 0 & 25 \end{pmatrix} \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 11\omega \\ -2\omega \\ 0 \end{pmatrix} \tag{34.15}$$

We see that the angular momentum has non zero x and y components, hence it is not along the x axis and hence it is not parallel to the angular velocity.

Case III : Angular velocity along the y axis In this case $\vec{\omega} = (0, \omega, 0)$ and the angular momentum is given by

$$\underline{L} = \underline{I} \cdot \underline{\omega} = \begin{pmatrix} 11 & -2 & 0 \\ -2 & 14 & 0 \\ 0 & 0 & 25 \end{pmatrix} \begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix} = \begin{pmatrix} -2\omega \\ 14\omega \\ 0 \end{pmatrix} \tag{34.16}$$

Again the angular momentum vector is not parallel to the angular velocity.

Lesson 35

Torque for constant angular velocity: Examples

Oct 31, 2012

Example 21: What is the moment of inertia tensor for a rectangular lamina of sides a and b.

Choose the centre of the plate as the origin and X, Y axes parallel to the sides as shown in the figure.

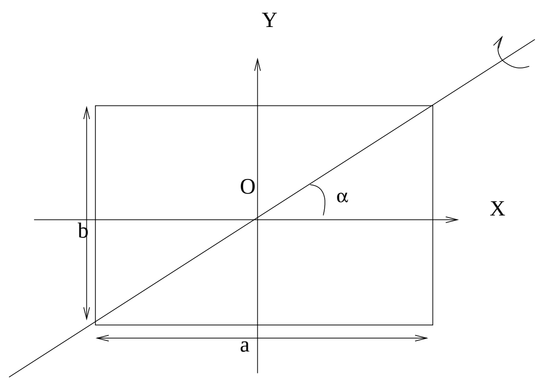

Fig. **36.** Figure for the Examples 1 and 2

We divide the plate into small small pieces and find contribution of each piece to the moment of inertia tensor components. So at first divide the lamina into a large number of thin vertical strips parallel to the Y axis. consider one such vertical strip between x and $x + dx$. We will sum over all such strips by integrating over x from $-a/2$ to $a/2$.

Next divide the strip at x into small small rectangular pieces by drawing horizontal lines parallel to the Y axis. One such rectangular element is shown as shaded. This element is located at $(x, y, 0)$ has an area of $dxdy$ and its mass will be $\sigma dxdy \equiv dm$ where σ is the mass per unit area. The total mass is M therefore $\sigma = M/ab$.The contribution to 11 component moment of the inertia tensor is $dm(y^2 + z^2)$ and the total value of I_{11} is obtained by summing (integrating) over all pieces. This is done by integrating over x from $-a/2$ to $a/2$ over y from $-b/2$ to $b/2$. So we have

$$\begin{aligned}
I_{xx} &= \int_{-a/2}^{a/2} \Big\{ \int_{-b/2}^{b/2} \sigma(y^2+0)dy \Big\} dx \quad (\because z=0) \\
&= \sigma \int_{-a/2}^{a/2} \Big\{ \frac{y^3}{3}\Big|_{-b/2}^{b/2} \Big\} dx \\
&= \sigma \int_{-a/2}^{a/2} \frac{b^3}{12} dx \\
&= \sigma \frac{b^3}{12} [x]\Big|_{-a/2}^{a/2} dx \\
&= \sigma \frac{b^3}{12}\, a \\
&= \frac{Mb^2}{12} \qquad (\because \sigma = M/(ab))
\end{aligned} \tag{35.1}$$

Remember a double integral is computed by evaluated the inner integral over y first Then integrating the result over the outside variable x. In this example the order of integration is unimportant.

Similarly $I_{22} = \dfrac{Ma^2}{12}$. Also, for this case (not in general) $I_{33} = I_{xx} + I_{yy}$. (Prove this yourself.) Next we compute I_12.

$$I_{xy} = -\int_{-a/2}^{a/2} \Big\{ \int_{-b/2}^{b/2} \sigma\, x\, y\, dy \Big\} dx \tag{35.2}$$

The inner integral, over y, is

$$\int_{-b/2}^{b/2} ydy = \frac{y^2}{2}\Big|_{-b/2}^{b/2} = \frac{b^2}{2} - \frac{b^2}{2} = 0. \tag{35.3}$$

Similarly other off diagonal components of the moment of inertia tensor turn out to be zero. This is because of our choice of the coordinate axes is symmetrical w.r.t. the plate.

Example 22: Find torque acting on a rectangular plate when it rotates with a constant angular velocity about

(a) the Z- axis

(b) about its diagonal. ($\hat{n} = \frac{1}{\sqrt{2}}(1, 1, 0)$.

The choice of axes is as in the previous example.

The moment of inertia tensor is give by

$$\underline{\mathsf{I}} = \frac{M}{12}\begin{pmatrix} b^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & (a^2+b^2) \end{pmatrix} \tag{35.4}$$

Using $\frac{d\omega}{dt} = 0$ in the Euler's equations

$$\frac{d\vec{L}}{dt} + \vec{\omega} \times \vec{L} = \vec{\tau}_{\text{total}} \tag{35.5}$$

will give the required torque. Now the components of angular momentum are given by

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \frac{M}{12}\begin{pmatrix} b^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & (a^2+b^2) \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \frac{M}{12}\begin{pmatrix} b^2\omega_x \\ a^2\omega_y \\ (a^2+b^2)\omega_z \end{pmatrix} \tag{35.6}$$

Note that the components of $\vec{L}$ are proportional to components of $om\vec{e}ga$ Hence $\frac{d\vec{\omega}}{dt} = 0$ means $\frac{d\vec{L}}{dt} = 0$. Therefore, The required torque is given by

$$\vec{\tau}_{\text{total}} = \omega \times \vec{L}. \tag{35.7}$$

(a) In the first case angular velocity and angular momentum components are given by

$$\vec{\omega} = (0, 0, \omega), \qquad \vec{L} = \frac{M}{12}(0, 0, (a^+ b^2)) \tag{35.8}$$

and required torque vanishes because $\vec{\omega}$ and $\vec{L}$ parallel (both being along the Z-axis.)

(b) In this case

$$\omega = \omega(\cos\alpha, \sin\alpha, 0), \qquad \vec{L}\frac{M\omega}{12}(b^2\cos\alpha, a^2\sin\alpha, 0) \tag{35.9}$$

where α is the angle the diagonal makes with the X- axis.

$$\vec{\omega} \times \vec{L} = \frac{M\omega^2}{12}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\alpha & \sin\alpha & 0 \\ b^2\cos\alpha & a^2\sin\alpha & 0 \end{vmatrix} \tag{35.10}$$

Note that both the vectors $\vec{L}$ and $\vec{\omega}$ lie in $X - Y$ plane therefore the cross product is along the Z- axis and the z component of the torque is given by

$$\tau_z = \frac{M\omega^2}{12}(a^2 - b^2)\sin\alpha\,\cos\alpha. \tag{35.11}$$

Thus the torque is seen to be non zero if $b \neq a$.

Lesson 36

Examples: Bicyle wheel, Kater's pendulum, Equilibrium

Oct 31, 2012

§1 Spinning bicycle wheel

We discuss the dynamics of a spinning wheel. You have already watched this video on the moodle course site. A string is tied to one of the ends, P, of axle of a bicycle wheel and is the other end is supported manually by holding it with hand. The string is kept vertical by, for example tying the other end C of the string to the ceiling. When the end held by hand is released, the wheel falls due to gravity. However, if the wheel is set spinning before being released, it does not fall; it precesses in one direction. The direction of precession gets reversed when if the wheel is set spinning in the opposite direction.

How do we understand this? Let the horizontal direction in which the axle points be chosen as X_2 axis so that the angular velocity and angular momentum points along the X_2. Let the X_3 axis be chosen upwards so that the force $F = mg$ due to gravity is along the negative X_2 direction. The radius vector $\vec{x}$ of the centre of mass of the wheel is along the positive X_2 axis. So that the torque, $\vec{x} \times \vec{F}$ is then along the negative X_1 axis.

The change in angular momentum in a short time time Δt is given by $\Delta\vec{L} = \vec{\tau}\Delta t$ and is shown in Fig.37(a). The vector sum of $\vec{L}$ and $\Delta\vec{L}$, drawn in Fig.37(b) is along PD. The implies that after time Δt the axle has rotated about the X_3 axis so that the end C moves to the point D.

If the wheel is initially set spinning in the opposite direction, the torque and hence the change in angular momentum $\Delta\vec{L}$ remains the same, but the direction of $\vec{L}$ is reversed and the angular momentum after time Δt will be as shown in Fig.37(c). This means that the end C must move to the point E and hence the axle rotates in the opposite direction.

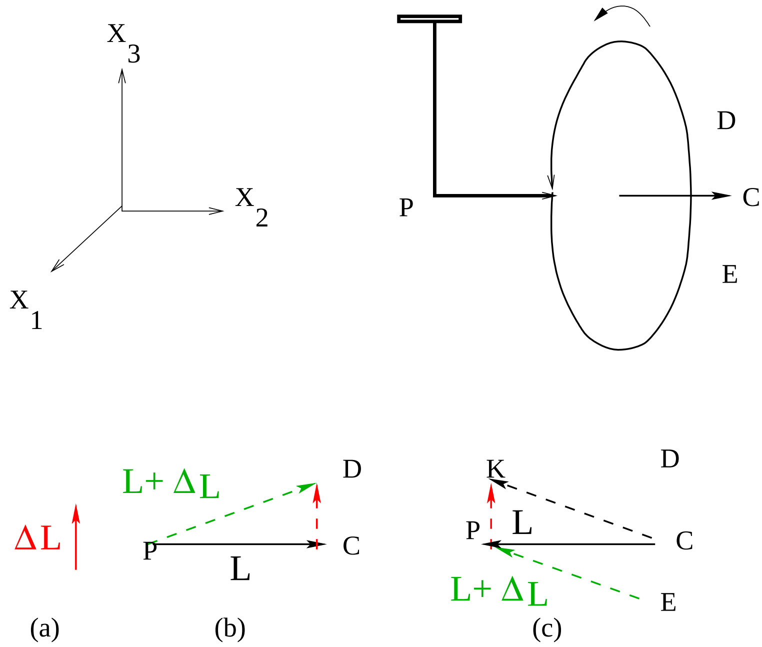

Fig. **37.** Movement of Spinning bicycle wheel

[Q/T] := The above analysis has been presented from the point of view of the space axes. Analyse the motion from the point of view of the body frame.

§2 Kater's pendulum

You would have done this experiment as part of mechanics laboratory. A rectangular bar having holes along its length. The time period of oscillations is measured by suspending it at different holes. This information is then used to compute the value of ion due to gravity.

Let the point of suspension of the Kater's pendulum be at a distance h from the centre of gravity. The moment of inertia about the point of suspension is

$$I = I_0 + Mh^2 \quad \text{where } I_0 = \frac{ML^2}{12} \tag{36.1}$$

where L is the length and M the mass of the pendulum bar. If the angular displacement at time t is θ, the angular velocity and the angular momentum are given by

$$\omega = \frac{d\theta}{dt}; L = I\omega. \tag{36.2}$$

We take the X_3 axis pointing outwards and perpendicular to the plane of paper. The angular velocity and angular momentum are along the positive X_3 axis. The torque of the the gravitational force is given by

$$\vec{\tau} = \vec{r} \times \vec{F}. \tag{36.3}$$

Remembering that the point of suspension must be taken as the origin, the magnitude of the torque is given by

$$\tau = |\vec{r}| mg \times \sin\theta = mgh \sin\theta. \tag{36.4}$$

Therefore the equation of motion is given by

$$\frac{dI\omega}{dt} = -mgh \sin\theta. \tag{36.5}$$

A minus sign appears because the direction of the torque is into the plaane of paper, opposite to the positive X_3 axis. For small angular amplitudes of oscillations, $\sin\theta \approx \theta$ and hence the equation of motion takes the form

$$\frac{dI\omega}{dt} = -mgh\theta \Rightarrow I\frac{d^2\theta}{dt^2} = -mgh\theta. \tag{36.6}$$

and the frequency ν and the time period T of oscillation are given by

$$\nu^2 = \frac{mgh}{I} \tag{36.7}$$

$$T = \frac{2\pi}{\nu} = 2\pi\sqrt{\frac{I}{mgh}}. \tag{36.8}$$

We use notation $K^2 = \frac{L^2}{12}$, where K is called the radius of gyration, and write the above relation for time period as

$$\frac{gT^2}{4\pi^2} = h + \frac{K^2}{h}. \tag{36.9}$$

For any given values of there are two values of h, say h_1, h_2. Then

$$h_1 + h_2 = \frac{gT^2}{4\pi^2}. \tag{36.10}$$

A graph of T^2 vs h is plotted and for a chosen value of T, we can determine h_1, h_2 and hence value of g can be computed. This gives a method for an accurate determination of g.

§3 Equilibrium of rigid bodies

The most general displacement of a rigid body is a translation and a rotation. The EOM separate into two parts. In an inertial frame we have

$$M\frac{d^2\vec{X}_{\rm cm}}{dt^2} = \vec{F}^{\rm tot}_{\rm ext}, \tag{36.11}$$

$$\frac{d\vec{L}}{dt} = \vec{\tau}^{\rm tot}_{\rm ext}. \tag{36.12}$$

The first equation describes the translational motion of the centre of mass. The second equation gives the rotation about the centre of mass. For a body to remains at rest it is necessary that the total external force be zero and the total torque of all external forces be zero. In general the value of the torque dependes on the choice of the origin. But if the total the resultant of a set of force is zero and if total torque is zero w.r.t. a point, the value of torque will be zero w.r.t. every any point chosen as origin. (WHY?) So the two necessary conditions for equlibrium of a rigid body are

$$\vec{F}^{\rm tot}_{\rm ext}, \quad \text{and} \quad \vec{\tau}^{\rm tot}_{\rm ext}. \tag{36.13}$$

$[Q/T] :=$ Are the two conditions sufficient also or not ? Give examples/reasons.

Example 23: Consider the equilibrium of a ladder resting on a rough floor and a frictionless wall as shown in the Fig.38.

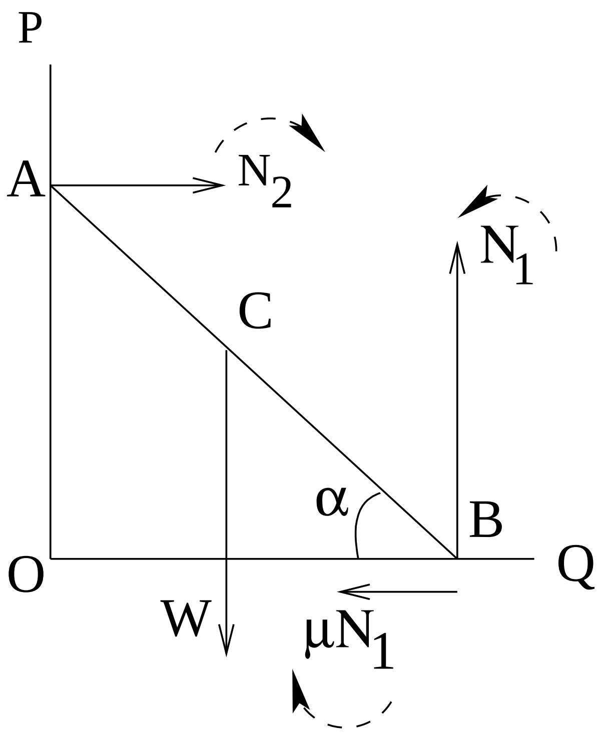

Fig. **38.** Forces on a Ladder

Let the reactions of the floor and the wall be N_1 and N_2 respectively. The frictional force on the ladder due to the floor will be μN_1 as shown in the figure. Equating the horizontal and vertical components of the forces to zero we get

$$N_2 = \mu N_1, \quad W = N_1. \tag{36.14}$$

We equate the sum of torques of all forces about the center of mass to zero. So chose the X_3 axis passing through the the centre of mass and assign out of the plane of the paper as its positive direction. The torques of different forces tend to rotate the ladder as shown. Assigning positive sign for anti clockwise and negative sign for clockwise rotations, we find that

$$\begin{aligned} \text{Torque of } N_1 &= -N_1 L \sin\alpha \times & (36.15)\\ \text{Torque of } N_2 &= -N_2 L \sin\alpha & (36.16)\\ \text{Torque of } \mu N_1 &= +\mu N_1 L \cos\alpha & (36.17)\\ \text{Torque of } W &= 0 & (36.18)\end{aligned}$$

where $2L$ is the length of the ladder assumed to be uniform. Equating the total torque to zero we get

$$N_1 L\mu\cos\alpha - N_1 L\sin\alpha - N_2\sin\alpha = 0. \tag{36.19}$$

Substituting $N_2 = \mu N_1$ in the above equation we gets

$$\mu\cos\alpha - \sin\alpha - \mu\sin\alpha = 0 \Rightarrow \tan\alpha = \frac{\mu}{\mu+1}. \tag{36.20}$$

[Q/T]:= A rectangular block kept on a rough inclined plane may not slide if friction is sufficiently large, but a cylinder on an inclined plane will always roll down.WHY?

[Q/T]:= Can a ladder resting against a smooth wall and rough floor remain in equilibrium if the angle with the horizontal is greater than 45^o?

Example 24: As another example of equilibrium consider a cylinder resting against a step as shown in Fig.39. Find the minimum force F, applied as shown in the figure, so that the cylinder may climb up the step.

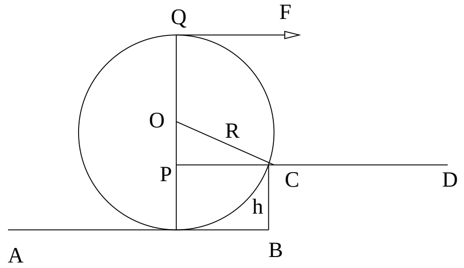

Fig. **39.** When does the cylinder climb the step?

Draw a free body diagram for the cylinder and considering the torques of all the forces about a suitable point, find the minimum required force for $h = 2R/5, \mu = 2/5$.

Lesson 37

Combined Translational and Rotational Motion

Nov 2, 2012

§1 Combined Rotational and Translational Motion

A example of a body undergoing a combined rotational and translational motion is a sphere rolling down an inclined plane. Two special extreme cases of interest are that of pure slipping and pure rolling. A general motion is involves both rolling and slipping.

For pure rolling, *i.e.* rolling without slipping, the frictional forces are small and can be neglected. So what is the condtion for pure rolling? Consider a sphere rolling on a horizintal plane. If the angular velocity of the sphere about the centre is ω, then the linear velocity for pure rolling is given by $v = \omega R$, where R is the radius of the sphere.

Example 25: More Examples Here

Example 26: A sphere of mass M and radius R, starting from rest, rolls down an inclined plane without slipping. Find the velocity acquired by it when it reaches the bottom of the plane.

Initial Kinetic energy of the sphere is zero and the potential energy $= Mgh$, where h is the height of the top point of plane. The final potential energy is zero and the kinetic energy is sum of rotational kinetic energy $\frac{1}{2}I\omega^2$ and the translational kinetic energy is $\frac{1}{2}MV^2$. Therefore, the conservation of energy gives

$$\frac{1}{2}MV^2 + \frac{1}{2}I\omega^2 = Mgh. \tag{37.1}$$

For pure rolling $V = \omega R$ and for a solid sphere $I = \frac{2}{5}MR^2$. This gives

$$\frac{1}{2}MV^2 + \frac{1}{5}MV^2 = Mgh \Rightarrow V^2 = \frac{10gh}{7}. \tag{37.2}$$

In general the sphere will slip for some distance and then only pure rolling will take place.

[Q/T] = How will the above analysis change if slipping takes place for some distance?

Example 27: A sphere resting on a frictionless plane is given a horizontal push at the level of the centre so that it acquires a velocity u. After travelling some distance it crosses onto a rough horizontal plane. Find the velocity of the when it slipping stops and pure rolling starts.

As long as the sphere is on fictionless part it slips without rolling (WHY?) and moves with uniform velocity. Assume that pure rolling starts at time t after it reaches the boundary separating the smooth and rough parts.

Now the only horizontal force on the sphere is frictional force F and it produces a torque $\tau = FR$. The equations of motion then give

$$F = Ma, \qquad \tau = I\alpha, \tag{37.3}$$

If the angular acceleration is α and linear deceleration is a, after time t the velocity and angular velocity are given by

$$v(t) = u - at; \qquad \omega(t) = \alpha t. \tag{37.4}$$

Substituting for a and α from EOM we get

$$v(t) = u - \frac{Ft}{M}, \qquad \omega(t) = \frac{\tau t}{I}. \tag{37.5}$$

Therefore solving for t we get

$$t = \frac{I\omega(t)}{\tau}. \tag{37.6}$$

If pure rolling starts at time t then at that time $v = \omega R$ and

$$t = \frac{Iv}{R\tau}. \tag{37.7}$$

substituting in the expression for velocity gives

$$v = u - \frac{Ft}{M} \tag{37.8}$$

$$\Rightarrow v = u - \frac{F}{M}\frac{vI}{R\tau} \tag{37.9}$$

$$\Rightarrow v = u - \frac{vI}{MR^2} \qquad (\because \tau = FR) \tag{37.10}$$

$$\Rightarrow v = u - \frac{2v}{5} \qquad (\because I = \frac{2}{5}MR^2) \tag{37.11}$$

Solving for v gives the required answer $v = \dfrac{5}{7}u$.

Lesson 38

Concluding Remarks

Nov 5, 2012

The basic concepts and methods in the rigid body dynamics have been outlined in the last few lectures. The topic of rigid body dynamics is extremely important for design of machines. Special attention has to be paid and care is required to work out, for example, how a moving part of a car or train will respond to the over all movement, specially to a rotatory motion, of the vehicle.

Time did not permit us to take up more examples. However, there are is new Physics to be learnt. We did not have time to discuss simple harmonic motion. This will be part of next course on Waves and Oscillations.

In the set of lectures on gravitation the some more time could have been spent on gravitational field and computation of gravitational potential. This could not be taken up meaningfully, as it requires an understanding of multiple integrals. Both the electrostatic and gravitational forces satisfy inverse square law and share many common properties. In fact computation of gravitational field and gravitational potential due to a mass distribution is not different that of electric field and electric potential due to a charge distribution. In the relation between the field and the potential, conservation of energy etc work exactly same way in the two cases.

You must learn to connect up similar concepts and methods of solution etc in different areas of and learn them in isolation. You must try to build upon what you have learnt where ever possible. Do not keep different things in separate water tight compartments instead try to discover connections between seemingly diverse topics you learn.