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Laplace-Runge-Lenz Vector

The potential for Coulomb problem is $-k/r$

$$\therefore \quad L = \frac{1}{2} m \dot{\vec{r}}^2 - \frac{k}{r}$$

Given transformation is

$$\vec{r} \to \vec{r}\,' = \vec{r} - m\left[2\dot{\vec{r}}(\vec{\epsilon}\cdot\vec{r}) - \vec{r}(\vec{\epsilon}\cdot\dot{\vec{r}}) - (\vec{r}\cdot\dot{\vec{r}})\epsilon\right]$$

To show that this transformation is a symmetry transformation we must compute δL and show that it is total time derivative.

(a) $\delta(K.E.) = \delta\left(\frac{1}{2} m \dot{\vec{r}}^2\right)$

$$= m\dot{\vec{r}}\left(\frac{d}{dt}\delta\vec{r}\right)$$

$$= -m^2\dot{\vec{r}}\Big[2\ddot{\vec{r}}(\vec{\epsilon}\cdot\vec{r}) + 2\dot{\vec{r}}(\epsilon\cdot\dot{\vec{r}}) - \dot{\vec{r}}(\vec{\epsilon}\cdot\dot{\vec{r}}) - \vec{r}(\vec{\epsilon}\cdot\ddot{\vec{r}}) - (\dot{\vec{r}}\cdot\dot{\vec{r}})\vec{\epsilon} - (\vec{r}\cdot\ddot{\vec{r}})\vec{\epsilon}\Big]$$

$$= -m^2\Big[2(\dot{\vec{r}}\cdot\ddot{\vec{r}})(\vec{\epsilon}\cdot\vec{r}) + \underline{2(\dot{\vec{r}}\cdot\dot{\vec{r}})(\vec{\epsilon}\cdot\dot{\vec{r}})} - \underline{(\dot{\vec{r}}\cdot\dot{\vec{r}})}\underline{(\vec{\epsilon}\cdot\dot{\vec{r}})} - (\dot{\vec{r}}\cdot\vec{r})(\vec{\epsilon}\cdot\ddot{\vec{r}}) - \underline{(\dot{\vec{r}}\cdot\dot{\vec{r}})}\underline{(\vec{\epsilon}\cdot\dot{\vec{r}})} - (\ddot{\vec{r}}\cdot\vec{r})(\vec{\epsilon}\cdot\dot{\vec{r}})\Big]$$

$$= -m^2\Big[2(\dot{\vec{r}}\cdot\ddot{\vec{r}})(\vec{\epsilon}\cdot\vec{r}) - (\vec{r}\cdot\ddot{\vec{r}})(\vec{\epsilon}\cdot\dot{\vec{r}}) - (\dot{\vec{r}}\cdot\vec{r})(\vec{\epsilon}\cdot\ddot{\vec{r}})\Big]$$

Runge Lenz Vector is

$$\vec{N} = \vec{p}\times\vec{L} - \frac{km}{r}$$

Variation of K.E. should be related to $\frac{d}{dt}(\vec{p}\times\vec{L})\cdot\vec{\epsilon}$

Compute

$$\vec{\epsilon}\cdot\frac{d}{dt}(\vec{p}\times\vec{L})$$

$$= \frac{d}{dt}\left(m^2\,\vec{\epsilon}\cdot\dot{\vec{r}}\times(\vec{r}\times\dot{\vec{r}})\right)$$

$$= m^2\left[\vec{\epsilon}\cdot\ddot{\vec{r}}\times(\vec{r}\times\dot{\vec{r}}) + \dot{\vec{r}}\times(\cancel{\dot{\vec{r}}\times\dot{\vec{r}}})^{0}\cdot\vec{\epsilon} + \vec{\epsilon}\cdot\dot{\vec{r}}\times(\vec{r}\times\ddot{\vec{r}})\right]$$

$$= m^2\left[\vec{\epsilon}\cdot\{(\ddot{\vec{r}}\cdot\dot{\vec{r}})\vec{r} - (\ddot{\vec{r}}\cdot\vec{r})\dot{\vec{r}}\} + \epsilon\{(\dot{\vec{r}}\cdot\ddot{\vec{r}})\vec{r} - (\dot{\vec{r}}\cdot\vec{r})\ddot{\vec{r}}\}\right]$$

$$= m^2\left[2(\vec{\epsilon}\cdot\vec{r})(\dot{\vec{r}}\cdot\ddot{\vec{r}}) - (\vec{\epsilon}\cdot\dot{\vec{r}})(\dot{\vec{r}}\cdot\vec{r}) - (\vec{\epsilon}\cdot\ddot{\vec{r}})(\vec{r}\cdot\dot{\vec{r}})\right]$$

$$\therefore\ \delta\left(\tfrac{1}{2}m\dot{\vec{r}}^2\right) = -\vec{\epsilon}\cdot\frac{d}{dt}(\vec{p}\times\vec{L})$$

Next we check relation between δ(P.E.) and $\frac{d}{dt}\left(-\frac{km}{r}\right)$.

(b) Compute variation of potential energy

$$\delta(P.E.) = -k\,\delta\left(\tfrac{1}{r}\right)$$

$$= -k\nabla\left(\tfrac{1}{r}\right)\cdot\delta\vec{r} \qquad \nabla(1/r) = -\frac{\vec{r}}{r^3}$$

$$= k\,\frac{\vec{r}}{r^3}(-m)\left[2\dot{\vec{r}}(\vec{\epsilon}\cdot\vec{r}) - \vec{r}(\vec{\epsilon}\cdot\dot{\vec{r}}) - (\vec{r}\cdot\dot{\vec{r}})\vec{\epsilon}\right]$$

$$= -\frac{km}{r^3}\left[2(\vec{r}\cdot\dot{\vec{r}})(\vec{\epsilon}\cdot\vec{r}) - (\vec{r}\cdot\vec{r})(\vec{\epsilon}\cdot\dot{\vec{r}}) - (\vec{r}\cdot\dot{\vec{r}})(\vec{\epsilon}\cdot\vec{r})\right]$$

$$= -\frac{km}{r^3}\left[(\dot{\vec{r}}\cdot\vec{r})(\vec{\epsilon}\cdot\vec{r}) - (\vec{r}\cdot\vec{r})(\vec{\epsilon}\cdot\dot{\vec{r}})\right]$$

$$= -\frac{km}{r^3}(\vec{\epsilon}\cdot\vec{r})(\dot{\vec{r}}\cdot\vec{r}) - \frac{km}{r}(\vec{\epsilon}\cdot\dot{\vec{r}})$$

$$\frac{d}{dt}\frac{km}{r}(\vec{\epsilon}\cdot\vec{r}) = -\frac{km}{r^3}(\vec{r}\cdot\dot{\vec{r}})(\vec{\epsilon}\cdot\vec{r}) + \frac{km}{r}(\vec{\epsilon}\cdot\dot{\vec{r}})$$

$$\delta(P.E.) = +\frac{d}{dt}\frac{km}{r}(\vec{\epsilon}\cdot\vec{r})$$

$$\therefore \delta L = \delta(K.E.) - \delta(P.E.)$$

$$= -\vec{\epsilon}\cdot\frac{d}{dt}\bar{p}\times\bar{L} - \frac{d}{dt}\vec{\epsilon}\cdot\left(\frac{km}{r}\vec{r}\right)$$

$$= -\vec{\epsilon}\,\frac{d}{dt}\left(\bar{p}\times\bar{L} + \frac{km\vec{r}}{r}\right) = -\frac{d}{dt}\Omega$$

$$\cancel{= -\vec{\epsilon}\cdot\frac{d}{dt}\vec{N}} \qquad \Omega = (\bar{p}\times\bar{L} + km\vec{r}/r)\cdot\vec{\epsilon}$$

∴ The given transformation is a symmetry transformation.

To show that the corresponding conserved quantity is Laplace-Runge-Lenz vector, recall Nother's theorem gives conservation law

$$\frac{d}{dt}G = 0 \qquad G = \sum p_k \delta q_k + \Omega$$

So compute

$$\sum p_k \delta q_k = \vec{p}\cdot\delta\vec{r}$$

$$= -m\,\vec{p}\,[2\vec{\dot{r}}(\vec{\epsilon}\cdot\vec{r}) - \vec{r}(\vec{\epsilon}\cdot\vec{\dot{r}}) - (\vec{r}\cdot\vec{\dot{r}})\vec{\epsilon}\,]$$

$$= -m^2\left[2(\vec{\dot{r}}\cdot\vec{\dot{r}})(\vec{\epsilon}\cdot\vec{r}) - (\vec{\dot{r}}\cdot\vec{r})(\vec{\epsilon}\cdot\vec{\dot{r}}) - (\vec{\epsilon}\cdot\vec{\dot{r}})(\vec{r}\cdot\vec{\dot{r}})\right]$$

$$= -2m^2\left[(\vec{\epsilon}\cdot\vec{r})(\vec{\dot{r}}\cdot\vec{\dot{r}}) - (\vec{\epsilon}\cdot\vec{\dot{r}})(\vec{r}\cdot\vec{\dot{r}})\right]$$

$$= -2m^2\vec{\epsilon}\cdot\left[(\vec{\dot{r}}\cdot\vec{\dot{r}})\vec{r} - (\vec{r}\cdot\vec{\dot{r}})\vec{\dot{r}}\,\right]$$

$$= -2m^2\vec{\epsilon}\cdot\left[\vec{\dot{r}}\times(\vec{r}\times\vec{\dot{r}})\right]$$

$$= -2\;\vec{\epsilon}(\vec{p}\times\vec{L})$$

$\therefore$ conserved quantity is

$$-2\,\vec{\epsilon}\cdot(\vec{p}\times\vec{L}) + \vec{\epsilon}\cdot(\vec{p}\times\vec{L}) + k\frac{m\vec{r}}{r}$$

$$= -\vec{\epsilon}\cdot\vec{N}$$

Since $\vec{\epsilon}$ is arbitrary, $\vec{N}$ is conserved.