Find the moment of inertia tensor of a uniform rectangular plate w.r.t. the centre O of the plate relative to the axes K shown in figure. Use parallel axes theorem to find moment of inertia tensor relative to a set of axes K_2 with origin taken as the corner E. The Z- axis for both systems is perpendicular to the plate and out of the plane of paper.

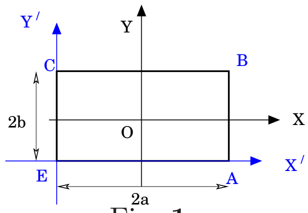

Fig. **1**

Usage Context Class Room Example, Quiz

☺ **Solution:** We shall do it by evaluating the double integral. Divide the plate by lines parallel to the coordinate axes as in Fig.**??**.

Consider one rectangular element of sides (dx, dy) at position (x, y). If the mass of the plate is M, the density of the plate is σ, then $4ab\sigma = M$ and the contribution of the rectangular element to I_{xx} is $\sigma(dx\,dy)y^2$. Summing over all elements means integrating over x, y over their respective ranges. Thus

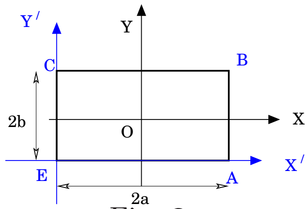

Fig. **2**

$$\begin{aligned} I_{xx} &= \int_{-a}^{a} dx \int_{-b}^{b} dy\sigma y^2 = \int_{-a}^{a} dx\sigma\frac{2b^3}{3} = \sigma\frac{2b^3}{3}\int_{-a}^{a} dx \\ &= \sigma\frac{2b^3}{3}(2a) = \frac{Mb^2}{3} \qquad \because 4\sigma ab = M \end{aligned} \tag{1}$$

Similarly

$$\begin{aligned} I_{yy} &= \int_{-a}^{a} dx \int_{-b}^{b} dy\sigma x^2 = \int_{-a}^{a} dx\sigma x^2(2b) = \sigma\frac{2b^3}{3}\int_{-a}^{a} dx \qquad \because 4\sigma ab = M \\ &= \sigma\frac{2a^3}{3}(2b) = \frac{Ma^2}{3} \end{aligned} \tag{2}$$

and appealing to the law of perpendicular axes we get $I_{zz} = M\frac{a^2+b^2}{3}$ The off diagonal term I_{xy} is

$$I_{xy} = -\int_{-a}^{a} dx \int_{-b}^{b} dy\sigma(xy) \tag{3}$$

which vanishes due to the symmetry of the problem. Also

$$I_{xz} = -\int_{-a}^{a} dx \int_{-b}^{b} dy\sigma(xz) = 0 \tag{4}$$

because $z = 0$ for all rectangular elements of the plate. Thus the moment of inertia tensor w.r.t.the center of mass is given by

$$\underline{\mathbb{I}} = \frac{M}{6}\begin{pmatrix} b^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{pmatrix} \tag{5}$$

MI Tensor relative to a corner The parallel axes for the inertia tensor states that

$$I_{jk} = I_{jk}^{\mathrm{cm}} + \Delta_{jk} \tag{6}$$

where $\Delta_{jk} = |\vec{a}|^2\delta_{jk} - a_j a_k$ and $\vec{a}$ is the position vector of the corner relative to the origin O. For the corner E we have $\vec{a} = (-a, -b, 0)$ and $|\vec{a}|^2 = (a^2+b^2)$. Therefore,

$$\Delta_{xx} = b^2, \qquad \Delta_{yy} = a^2, \qquad \Delta_{zz} = (a^2 + b^2) \tag{7}$$

$$\Delta_{xy} = \Delta_{yx} = -ab \qquad \Delta_{yz} = \Delta_{zy} = 0 \qquad \Delta_{zx} = \Delta_{xz} = 0. \tag{8}$$

Thus the moment of inertia tensor w.r.t.the corner at E is given by

$$\underline{\mathbb{I}} = \frac{M}{3}\begin{pmatrix} b^2 + 3a^2 & -3ab & 0 \\ -3ab & a^2 + 3b^2 & 0 \\ 0 & 0 & 4a^2 + 4b^2 \end{pmatrix}. \tag{9}$$

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PROOFS

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