☹ me-que-14004

Three equal masses, connected by light rods, are placed at the vertices of a right angle triangle as shown in figure. The origin of the coordinate system is chosen to coincide with the centre of mass of the system. Obtain the values of moment of inertia tensor components.

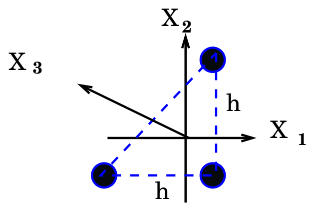


Fig. **1**

## Solution

**Method 1** Let the coordinates of the vertices $A, B, C$ be $(a, b, 0)$, $(a + h, b, 0)$ and $(a + h, b + h, 0)$. Since the centre of mass is at the origin and all the masses are equal

$$\begin{aligned} a + (a + h) + (a + h) = 0 &\Longrightarrow a = -2h/3 \\ b + b + b + h = 0 &\Longrightarrow b = -h/3. \end{aligned} \tag{1}$$

Therefore the coordinates of the points $A, B, C$ are

$$\vec{x}_1 = (-2h/3, -h/3, 0), \vec{x_2} = (h/3, -h/3, 0), \vec{x_3} = (h/3, 2h/3).$$

The moment of inertia tensor is given by

$$I_{jk} = \sum m_\alpha (|\vec{x}_\alpha|^2 \delta_{jk} - a_j a_k). \tag{2}$$

It is obvious that $I_{13} = I_{31} = I_{23} = I_{32} = 0$. Since the inertia tensor is symmetric we need to compute only $I_{11}, I_{22}, I_{33}$ and $I_{12}$. Let us write the values needed in a tabular form

| Location | $x_1$ | $x_2$ | $x_3$ | $x_1^2$ | $x_2^2$ | $\|\vec{x}\|^2$ | $x_1 x_2$ |
|---|---|---|---|---|---|---|---|
| $A$ | $-\frac{2h}{3}$ | $-\frac{h}{3}$ | 0 | $\frac{4h^2}{9}$ | $\frac{h^2}{9}$ | $\frac{5h^2}{9}$ | $\frac{2h^2}{9}$ |
| $B$ | $\frac{h}{3}$ | $-\frac{h}{3}$ | 0 | $\frac{h^2}{9}$ | $\frac{h^2}{9}$ | $\frac{2h^2}{9}$ | $-\frac{h^2}{3}$ |
| $C$ | $\frac{h}{3}$ | $\frac{2h}{3}$ | 0 | $\frac{h^2}{9}$ | $\frac{4h^2}{9}$ | $\frac{5h^2}{9}$ | $\frac{2h^2}{9}$ |
| $\sum_\alpha(.)$ | ... | ... | ... | $\frac{6h^2}{9}$ | $\frac{6h^2}{9}$ | $\frac{12h^2}{9}$ | $\frac{3h^2}{9}$ |

(3)

We are now ready to compute the moment of inertia tensor

$$I_11 = m\sum_\alpha\{|x_\alpha|^2 - x_{\alpha 1}x_{\alpha 1}\} = m(\frac{12h^2}{9} - \frac{6h^2}{3} = \frac{2mh^2}{3}) = \frac{2mh^2}{3} \quad (4)$$

$$I_22 = m\sum_\alpha\{|x_\alpha|^2 - x_{\alpha 2}x_{\alpha 2}\} = m\left(\frac{12h^2}{9} - \frac{6h^2}{3}\right) = \frac{2mh^2}{3} = \quad (5)$$

$$I_{33} = m\sum_\alpha\{|x_\alpha|^2 - x_{\alpha 3}x_{\alpha 3}\} = \frac{12mh^2}{9} - 0 = \frac{4mh^2}{3} \quad (6)$$

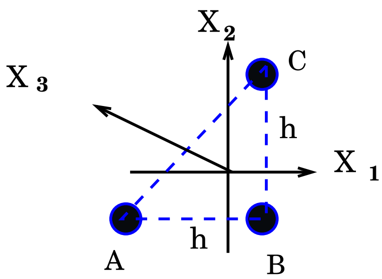

Fig. **2** $K'$ axes

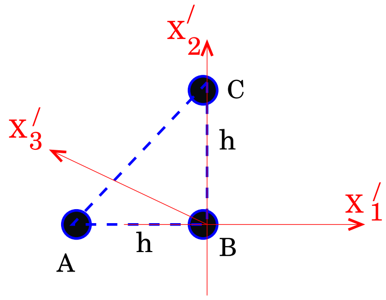

Fig. **3** $K'$ axes

**Method 2 ::Choose a convenient set of axes:**

We translate the axes from the given origin to vertex $B$, as shown in Fig.3. With respect to prime axes, the coordinates of the three vertices $A, B, C$ are $(-h, 0, 0), (0, 0, 0)$ and $(h, 0, 0)$. Therefore the components of moment of inertia tensor are given by

$$I_{11}{}' = mh^2, \quad I_{22} = mh^2, \quad I_{33} = 2mh^2. \quad (7)$$

All other components are zero.

$$I'_{12} = I'_{21} = 0, \quad I'_{23} = I'_{32} = 0, \quad I'_{31} = I'_{13} = 0. \quad (8)$$

Now we use parallel axes theorem to get the moment of inertia tensor components w.r.t. the given set with origin at the centre of mass. The coordinates of centre of mass in $K'$ system are given by $\vec{a} = (h/3, h/3, 0)$, and the total mass is $M = 3m$.
To use the parallel axes theorem

$$I_{jk} = I'_{jk} - M(|a|^2 - a_j a_k), \quad (9)$$

we record the following values computation of inertia tensor.

$$|\vec{a}|^2 = \frac{2h^2}{9} \qquad |a|^2 - a_1^2 = |a|^2 - a_2^2 = \frac{h^2}{9}, \qquad a_1 a_2 = \frac{h^2}{9}, \qquad a_1 a_3 = a_2 a_3 = 0$$

We are now ready to have the answers

$$\begin{aligned}
I_{11} &= I'_{11} - M(|a|^2 - a_1^2) = mh^2 - (3m)\frac{h^2}{9} = \frac{2mh^2}{3} \\
I_{22} &= I'_{22} - M(|a|^2 - a_2^2) = mh^2 - (3m)\frac{h^2}{9} = \frac{2mh^2}{3} \\
I_{33} &= I'_{33} - M(|a|^2 - a_3^2) = 2mh^2 - 3m\frac{2h^2}{9} = \frac{4mh^2}{3} \\
I_{12} &= I'_{12} + M(|a|^2 - a_1 a_2) = -(3m)\,\frac{h^2}{9} = -\frac{mh^2}{3} \\
I_{21} &= I_{12} = -(3m)\frac{h^2}{9} = -\frac{mh^2}{3}.
\end{aligned}$$

All other off diagonal components are zero. Thus the inertia tensor is given by

$$I = \begin{pmatrix} \frac{2mh^2}{3} & -\frac{mh^2}{3} & 0 \\ -\frac{mh^2}{3} & \frac{2mh^2}{3} & 0 \\ 0 & 0 & \frac{4mh^2}{3} \end{pmatrix}. \tag{10}$$