22.6.2004

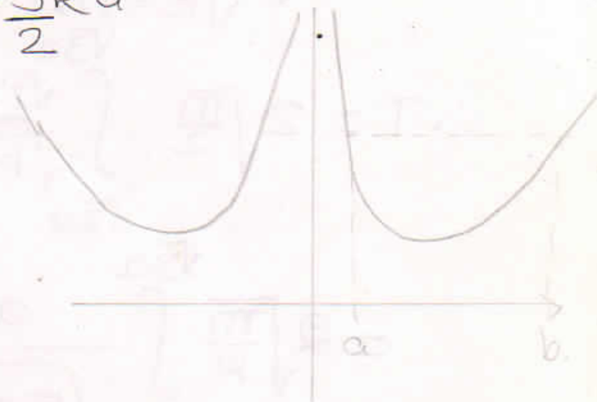
CM004

Equations of motion, Energy Conservation, one dimension

A particle in one dimension is subject to a force

$$F = -kx + \lambda/x^3$$

Find the potential energy and the exact period of oscillations for $\lambda = 6ka^4$, $E = \frac{5}{2}ka^2$



$F(x) = -kx + \lambda/x^3$

$V(x) = -\int F(x)dx$

$= \frac{1}{2}kx^2 + \frac{\lambda}{2x^2} + \text{const}$

choose reference pt so that const $\to 0$.

$V(x) = \frac{1}{2}kx^2 + \frac{\lambda}{2x^2}$

$$\boxed{\begin{aligned} V(a) &= \frac{k}{2}a^2 + \frac{\lambda}{2a^2} \\ &= \frac{7}{2}ka^2 \quad \text{Const}=0 \end{aligned}}$$

Turning points are given by

$\frac{5}{2}ka^2 = \frac{1}{2}kx^2 + \frac{3k}{x^2}a^4$

$x^2 + 6\frac{a^4}{x^2} - 5a^2 = 0.$

$x^4 - 5a^2x^2 + 6a^4 = 0$

$(x^2 - 3a^2)(x^2 - 2a^2) = 0 \quad x = \pm\sqrt{3}a, \pm\sqrt{2}a$

If $x > 0$ the motion takes place between $\sqrt{2}a$ and $\sqrt{3}a$.

CM004

Energy conservation

$$E = \frac{1}{2} m\dot{x}^2 + V(x)$$

$$\sqrt{\frac{m}{2}}\,\frac{dx}{dt} = \sqrt{E - V(x)}$$

$$\sqrt{\frac{m}{2}} \int \frac{dx}{\sqrt{E - V(x)}} = \int dt$$

$$\therefore T = 2\sqrt{\frac{m}{2}} \int_{\sqrt{2}a}^{\sqrt{3}a} \frac{\sqrt{2}x\,dx}{\sqrt{k}\sqrt{5a^2x^2 - x^4 - 6a^2}}$$

$$= 2\sqrt{\frac{m}{k}} \int_{\sqrt{2}a}^{\sqrt{3}a} \frac{x\,dx}{\sqrt{5a^2x^2 - x^4 - 6a^2}}$$

$$= 2\sqrt{\frac{m}{k}} \int_{2a^2}^{3a^2} \frac{dt}{(5a^2t - t^2 - 6a^2)^{1/2}}$$

$$= 2\sqrt{\frac{m}{k}} \int_{-1/2\,a^2}^{1/2\,a^2} \frac{dy}{(a^2/4 - y^2)^{1/2}}$$

$$= 2\sqrt{\frac{m}{k}} \int_{-1}^{+1} \frac{du}{(1-u^2)^{1/2}}$$

$$= 2\pi\sqrt{\frac{m}{k}}$$

$$\therefore\ \omega = \sqrt{\frac{k}{m}} \qquad T = 2\pi\sqrt{\frac{m}{k}}$$

$$E - V(x) = \frac{5}{2}ka^2 - \frac{1}{2}kx^2 - \frac{3ka^4}{x^2}$$

$$= \frac{1}{2}k\left(5a^2 - x^2 - \frac{6a^4}{x^2}\right)$$

$$= \frac{1}{2}\frac{k}{x^2}\left(5a^2x^2 - x^4 - 6a^4\right)$$

$$t^2 + 6a^2 - 5a^2t$$

$$\left(t - \frac{5}{2}a^2\right)^2 + 6a^2 - \frac{25}{4}a^2$$

$$= \left(t - \frac{5}{2}a^2\right) - \frac{a^2}{4}$$

$$y = t - \frac{5}{2}a^2$$

$\boxed{12\text{ pts}}$