QM-20 Spin and Identical Particles*

Solved problem

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Abstract

This solved problem explains construction of spin half matrices. The same procedure can be followed for arbitrary spin.

Problem 1: Construct the spin matrices for a spin half particle in the basis in which S_z is diagonal.

Here it is useful to recall from vector spaces that in an orthonormal basis $\{|k\rangle, k = 1, 2, .., n\}$ *the matrix for an operator* $\hat{T}$ *in Dirac notation is given by*

$$\boxed{\underline{\mathsf{T}} = \begin{pmatrix} \langle 1|\hat{T}|1\rangle & \langle 1|\hat{T}|2\rangle & \ldots & \langle 1|\hat{T}|n\rangle \\ \langle 2|\hat{T}|1\rangle & \langle 2|\hat{T}|2\rangle & \ldots & \langle 2|\hat{T}|n\rangle \\ \ldots & \ldots & \ldots & \ldots \\ \langle n|\hat{T}|1\rangle & \langle n|\hat{T}|2\rangle & \ldots & \langle n|\hat{T}|n\rangle \end{pmatrix}} \tag{1}$$

☺*Solution*: For a spin $\frac{1}{2}$ particle, the S_z eigenvalues are $\frac{\hbar}{2}, -\frac{\hbar}{2}$. In the basis chosen $S - z$ is diagonal therefore it is straightforward to write the matrix for S_z

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2}$$

Next we will need to recall

$$S_{\pm}|m\rangle = \sqrt{s(s+1) - m(m \pm 1)}\hbar|m \pm 1\rangle$$

and use it for $s = \frac{1}{2}, m = \pm\frac{1}{2}$ to get

$$S_+|\tfrac{1}{2}\rangle = 0 \quad \Rightarrow \quad \langle\tfrac{1}{2}|S_+|\tfrac{1}{2}\rangle = 0, \qquad \langle -\tfrac{1}{2}|S_+|\tfrac{1}{2}\rangle = 0 \tag{3}$$

$$S_+| -\tfrac{1}{2}\rangle = |\tfrac{1}{2}\rangle \quad \Rightarrow \quad \langle\tfrac{1}{2}|S_+| -\tfrac{1}{2}\rangle = 1, \qquad \langle -\tfrac{1}{2}|S_+| -\tfrac{1}{2}\rangle = 0 \tag{4}$$

$$\tag{5}$$

Therefore, the matrix

$$\begin{pmatrix} \langle\tfrac{1}{2}|S_+|\tfrac{1}{2}\rangle & \langle -\tfrac{1}{2}|S_+|\tfrac{1}{2}\rangle \\ \langle\tfrac{1}{2}|S_+| -\tfrac{1}{2}\rangle & \langle\tfrac{1}{2}|S_+|\tfrac{1}{2}\rangle \end{pmatrix} \tag{6}$$

* qm-ymp-20002 LUPDate: Oct 26, 2014

for S_+ becomes

$$S_+ = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{7}$$

The matrix for S_- is hermitian adjoint of S_+ and hence we get

$$S_- = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{8}$$

The matrices for, S_x, S_y, are therefore given by

$$S_x \quad = \tfrac{1}{2}\Big(S_+ + S_-\Big) = \quad \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{9}$$

$$S_y \quad = \tfrac{1}{2i}(S_+ - S_-) = \quad \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{10}$$

To summarize, we get the result that the spin $\frac{1}{2}$ matrices are given by

$$S_x = \frac{\hbar}{2}\sigma_1; \quad S_y = \frac{\hbar}{2}\sigma_2; \quad S_z = \frac{\hbar}{2}\sigma_3. \tag{11}$$

where $\sigma_1, \sigma_2, \sigma_3$ are Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{12}$$