☹ em-ymp-02002 fname

A conical surface (an empty ice cream cone) carries a uniform a uniform surface charge density σ_0. The height of the cone and the radius of the base are both equal to a. Find the potentials at the vertex of the cone and at the center of the base.

☺*Solution*: We divide the conical surface into thin rings by planes parallel to the base. Consider one such ring lying between planes at distances x and $x + dx$ from the vertex of the cone. Let 2α be the angle of the cone.

The the radius of the ring $= x \tan \alpha$.
The slant height of the ring $= \sqrt{2}dx$.
$\therefore$ area of the ring $= 2\pi x\sqrt{2}dx$
and the total charge on the ring $=2\pi x\sqrt{2}dx\sigma_0$. Potential at the vertex, P, due to this ring shaped element of area, is

$$d\phi(P) = \frac{1}{4\pi\epsilon_0} \times \sigma_0 2\pi x\sqrt{2}dx \times \frac{1}{\sqrt{2}x} \qquad (1)$$
$$= \frac{\sigma}{2\varepsilon_0} dx \qquad (2)$$

Hence the potential due to the cone at the vertex,P, is

$$\phi(P) = \int_0^a \left(\frac{\sigma}{2\epsilon_0}\right) dx \qquad (3)$$
$$= \left(\frac{\sigma a}{2\varepsilon_0}\right) \qquad (4)$$

The potential at the center of the base,Q, is due to the ring element at x is

$$d\phi(Q) = \left(\frac{1}{4\pi\varepsilon_0}\right) 2\pi x\sqrt{2}\, dx\sigma_0 \times \frac{1}{\sqrt{x^2 + (a-x)^2}} \qquad (5)$$
$$= \frac{\sqrt{2}\sigma_0}{2\varepsilon_0} \frac{x\, dx}{\sqrt{2x^2 - 2ax + a^2}} \qquad (6)$$

$\therefore$ the potential at Q is

$$\phi(Q) = \frac{\sqrt{2}\sigma_0}{2\varepsilon_0} \int_0^a \frac{x\, dx}{\sqrt{2x^2 - 2ax + a^2}} \qquad (7)$$
$$\equiv \frac{\sigma}{2\varepsilon_0} I \qquad (8)$$

where I is the integral given by

$$
\begin{aligned}
I &= \int_0^a \frac{x\,dx}{\sqrt{x^2 - ax + a^2/2}} && (9)\\
&= \int_0^a \frac{(x-a/2)\,dx}{\sqrt{x^2 - ax + a^2/2}} + \frac{a}{2}\int_0^a \frac{dx}{\sqrt{x^2 - ax + a^2/2}} && (10)\\
&= \sqrt{x^2 - ax + a^2/2}\Big|_0^a + \frac{a}{2}\int_0^a \frac{dx}{\sqrt{(x-a/2)^2 + a^2/4)}} && (11)\\
&= \frac{a}{2}\sinh^{-1}\left(\frac{(x-a/2)}{a/2}\right)\Big|_0^a && (12)\\
&= a\sinh^{-1}(1) && (13)
\end{aligned}
$$

<u>REMARKS</u> The integral appearing in Eq(11) can also be written as

$$
\begin{aligned}
\int_0^a \frac{dx}{\sqrt{(x-a/2)^2 + a^2/4)}} &= \int_{-a/2}^{a/2} \frac{dt}{\sqrt{t^2 + a^2/4}} && (14)\\
&= \log\left(t + \sqrt{t^2 + a^2/4}\right)\Big|_{-a/2}^{a/2} && (15)\\
&= \log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) && (16)\\
&= 2\log(\sqrt{2}+1) && (17)
\end{aligned}
$$