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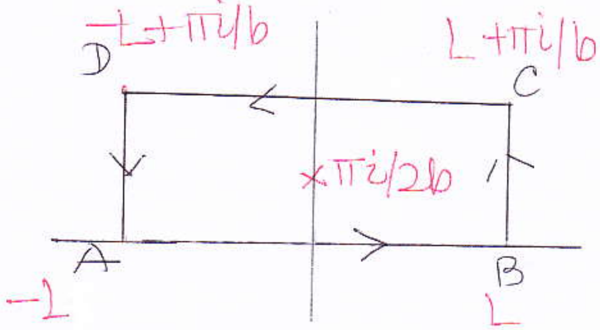
**Q 11**
**§§7.9** Compute integral $\int_0^\infty \frac{x \sinh a}{\cosh bx} dx$ using a ~~hyperb~~ rectangular contour. $(b > a > 0)$

First we note that

$$\int_0^\infty \frac{x \sinh ax}{\cosh bx} dx$$

$$= \int_{-\infty}^\infty \frac{x e^{ax}}{\cosh bx} dx$$




We integrate $f(z) = \frac{z e^{az}}{\cosh bz}$ around the rectangular contour ABCDA with corners at $-L$, $L$, $L+\pi i/b$, and $-L+\pi i/b$.

$$\oint f(z)dz = \int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CD} f(z)dz + \int_{DA} f(z)dz$$

The integrals along vertical sides DA and BC tend to zero as $L \to \infty$. In this limit

$$\lim \oint f(z)dz = \int_{-\infty}^\infty f(x)dx - \int_{-\infty}^\infty f(x+\pi i/b)\, dx$$

$$= \int_{-\infty}^\infty \frac{x e^{ax}}{\cosh bx} dx - \int_{-\infty}^\infty \frac{(x+\pi i/b)\, e^{ax+\pi ia/b}}{-\cosh bx} dx$$

$$= \int_{-\infty}^\infty \frac{(\pi i/b) e^{ax}}{\cosh bx} dx \; e^{\pi ia/b} + (1+e^{\pi ia/b}) \int_{-\infty}^\infty \frac{x e^{ax}}{\cosh bx} dx \quad \text{---(1)}$$

The contour ABCDA encloses only one simple pole of $f(z)$ at $z = i\pi/2b$. Hence

$$\oint f(z)dz = 2\pi i \times \operatorname{Res} \frac{z.e^{az}}{\cosh bz}\Big|_{z=\pi i/2b}$$

$$= 2\pi i \lim_{z \to \frac{\pi i}{2b}} z e^{az} \times \left(\frac{z - \pi i/2b}{\cosh bz}\right)$$

$$= (2\pi i)\left(\frac{\pi i}{2b}\right) e^{\pi i a/2b} \times \frac{1}{b \ \sinh(\pi i/2)}$$

$$= -\frac{\pi^2}{b} e^{\pi i a/2b} \times \frac{1}{ib}$$

$$= \frac{i\pi^2}{b^2} \exp(i\pi a/2b)$$

Substituting the value of integral in (1)

$$\frac{\pi i}{b} e^{ia\pi/b} \int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh bx} dx + (1 + e^{i\pi a/b}) \int_{-\infty}^{\infty} \frac{x e^{ax}}{\cosh bx} dx$$

$$= -\frac{i\pi^2}{b^2} \exp(\pi i a/2b)$$

$$\frac{i\pi}{b} e^{i\pi a/2b} \int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh bx} dx + 2\cos(\pi a/2b) \int_{-\infty}^{\infty} \frac{x e^{ax}}{\cosh bx} dx$$

$$= i\pi^2/b^2$$

Equating imaginary parts gives

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh bx} = \frac{\pi}{b} \sec(\pi a/b)$$

Equating real parts of the LHS to zero gives

$$-\frac{\pi}{b} \cancel{\cos} \sin\left(\frac{\pi i a}{b}\right) \int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh bx} dx + 2\cos\left(\frac{\pi a}{2b}\right) \int_{-\infty}^{\infty} \frac{x e^{ax}}{\cosh bx} dx = 0$$

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$$\therefore \int_{-\infty}^{\infty} \frac{x\, e^{ax}}{\cosh bx}\, dx = \frac{\pi}{2b} \tan\left(\frac{\pi a}{2b}\right) \int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh bx}\, dx$$

$$= \frac{\pi^2}{b^2} \sin\left(\frac{\pi a}{2b}\right) \sec^2(\pi a/2b)$$

or $$\int_0^{\infty} \frac{x \sinh ax}{\cosh bx}\, dx = \frac{\pi^2}{4b^2} \sin\left(\frac{\pi a}{2b}\right) \sec^2\left(\frac{\pi a}{2b}\right)$$

$$\therefore \int_{-\infty}^{\infty} \frac{x\, e^{ax}}{\cosh bx}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{x \sinh ax}{\cosh bx}\, dx$$

$$= 2\int_0^{\infty} \frac{x \sinh ax}{\cosh bx}\, dx$$