11.10.2017

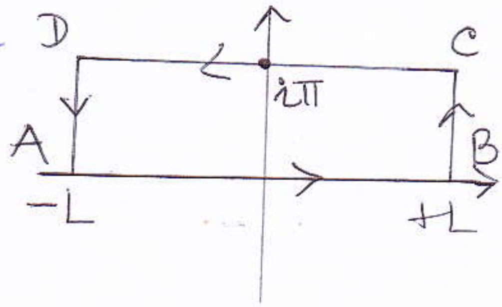
p1/Q1/§§7.9

Q[1] §§7.9 Compute the integral

$$\int_{-\infty}^{\infty} \frac{x e^{-x}}{1+e^{-4x}} dx$$

using contour integration around a rectangular contour.

Taking the rectangular contour ABCD of Fig 1, and integrating $f(z) = z e^{-z}/(1+e^{-4z})$ we see that



$$\oint_{ABCDA} f(z)dz = \int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CD} f(z)dz + \int_{DA} f(z)dz.$$

The integrals along BC and DA vanish as $L \to \infty$. (use Darboux theorem).

Therefore, in the limit $L \to \infty$,

$$\oint_{ABCDA} f(z)dz = \int_{AB} f(z)dz - \int_{DC} f(z)dz$$

$$= \int_{-L}^{L} f(x)dx - \int_{-L}^{L} f(x+i\pi)dx$$

$$= \int_{-L}^{L} \{f(x) - f(x+i\pi)\} dx$$

$$= \int_{-L}^{L} \left\{ \frac{x e^{-x}}{1+e^{-4x}} - \frac{(x+i\pi)\, e^{-x} e^{-i\pi}}{(1+e^{-4x})} \right\} dx$$

We have used $z = x$ for AB & $z = x+i\pi$ along DC.

$$= \int_{-L}^{L} \frac{2x\, e^{-x} dx}{1+e^{-4x}} + \int_{-L}^{L} \frac{i\pi e^{-x}}{1+e^{-4x}} dx$$

Hence

$$\int_{-\infty}^{\infty} \frac{2x\, e^{-x}}{1+e^{-4x}} dx + i\pi \int \frac{e^{-x}}{1+e^{-4x}} dx = \oint f(z)\,dz \quad \cdots (1)$$

For the functions $f(z) = \dfrac{z e^{-z}}{1+e^{-4z}}$ poles are ~~given by~~ located at

$$z = \pm \frac{i\pi}{4}, \pm \frac{3i\pi}{4}, \pm \frac{5\pi i}{4} \qquad \left(e^{4z} = -1\right)$$

Of these, only poles at $z = i\pi/4$, $3i\pi/4$ are enclosed by the contour ABCDA.

$$\text{Res}\{f(z)\}_{z=i\pi/4} = \left[(z - i\pi/4)\frac{z e^{-z}}{1+e^{-4z}}\right]_{z\to i\pi/4}$$

$$= z e^{-z}\Big|_{z=i\pi/4} \times \left[\frac{z - i\pi/4}{1+e^{-4z}}\right]_{z\to i\pi/4}$$ Use $\lim(f\cdot g) = (\lim f)(\lim g)$

$$= \frac{i\pi}{4} e^{-i\pi/4}\left[\frac{1}{-4e^{-4z}}\right]_{z\to i\pi/4}$$ Use L'Hopital's rule

$$= \frac{i\pi}{16} e^{-i\pi/4} \quad \#$$

$$\text{Res}\{f(z)\}_{z=\frac{3\pi i}{4}} = \left[\left(z - \frac{3\pi i}{4}\right)\frac{z e^{-z}}{1+e^{-4z}}\right]_{z\to \frac{3\pi i}{4}}$$

$$= z e^{-z}\Big|_{z=3\pi i/4} \times \left[\frac{z - 3\pi i/4}{1+e^{-4z}}\right]_{z\to 3\pi i/4}$$

$$= \frac{3\pi i}{4} e^{-3\pi i/4}\left(\frac{1}{-4e^{-3\pi i}}\right) = \frac{3\pi i}{16} e^{-3\pi i/4}$$

$$\oint f(z)\,dz = 2\pi i \times \left(\frac{i\pi}{16} \times \frac{1}{\sqrt{2}}(1+i) + \frac{3\pi i}{16} \times \frac{1}{\sqrt{2}}(-1+i) \right)$$

$$= \frac{-2\pi^2}{16\sqrt{2}} \times (1 - i - 3 - 3i)$$

$\sin\left(-\frac{3\pi}{4}\right) = -1$

$$= \frac{\pi^2}{8\sqrt{2}}(2 + 4i)$$

Substituting in Eq (1) and equating real and imaginary parts we get

$$\int_{-\infty}^{\infty} \frac{x e^{-x}}{1+e^{-4x}}\,dx = \frac{\pi^2}{8\sqrt{2}}$$

$$\int_{-\infty}^{\infty} \frac{e^{-x}}{1+e^{-4x}}\,dx = \frac{\pi}{2\sqrt{2}}.$$