22.9.2017

Q9
§§7.7

Integrate $\int_0^\infty \frac{\log x \, dx}{(x^2+1)(1+a^2x^2)}$ using technique of contour integration.

We will integrate $f(z) = \frac{\text{Log}\, z}{(z^2+1)(1+a^2z^2)}$ around closed contour $\Gamma$ of Fig 1 and take limit $R \to \infty$, $\varepsilon \to 0$.

Log z is the principal value of logarithm function.

Darboux theorem gives

$$\int_{BCA} f(z)\, dz \to 0 \text{ as } R \to \infty.$$

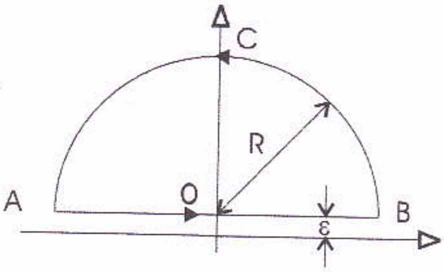


Fig. 1 Contour Γ

Therefore

$$\lim \oint_\Gamma f(z)\,dz = \int_{AO} f(z)\,dz + \int_{OB} f(z)\,dz$$

$$= \int_{OB} \frac{\text{Log}\, z \; dz}{(z^2+1)(z^2a^2+1)} - \int_{OA} \frac{\text{Log}\, z \; dz}{(z^2+1)(1+a^2z^2)}$$

Set up the two integrals using

along OB:

$z = r, \; 0 < r < R$

$dz = dr$

$$\frac{\text{Log}\, z}{(z^2+1)(1+a^2z^2)} = \frac{\ln r}{(r^2+1)(1+a^2r^2)}$$

along OA

$z = -r \quad 0 < r < R$

$dz = -dr$

$$\frac{\text{Log}\, z}{(z^2+1)(1+a^2z^2)} = \frac{\ln r + i\pi}{(r^2+1)(1+a^2r^2)}$$

Hence

$$\lim \oint_\Gamma f(z)\,dz = 2\int_0^\infty \frac{\ln r \; dr}{(r^2+1)(1+a^2r^2)} + 2\pi \int_0^\infty \frac{dr}{(r^2+1)(1+a^2r^2)} \quad (1)$$

To compute the integral along $\Gamma$, we use the residue theorem. Compute residues at $z = i, i/a$, the other poles are in lower half plane and do not contribute to the integral. (Rem $a>0$)

$$\operatorname{Res}\{f(z)\}_{z=i} = \lim_{z\to i} \frac{(z-i)\operatorname{Log} z}{(z^2+1)(1+a^2z^2)}$$

$$= \lim_{z\to i} \frac{\operatorname{Log} z}{(z+i)(1+a^2z^2)} = \left(\frac{i\pi}{2}\right)\times\frac{1}{2i}\times\frac{1}{(1-a^2)}$$

$$= \frac{\pi}{4}\frac{1}{(1-a^2)}$$

$$\operatorname{Res}\{f(z)\}_{z=i/a} = \lim_{z\to i/a} \frac{(z-i/a)\operatorname{Log} z}{(z^2+1)\,a^2(z^2+1/a^2)}$$ || Care needed here

$$= \lim_{z\to i/a} \frac{\operatorname{Log} z}{(z^2+1)(z+i/a)\,a^2} = \frac{\ln a + i\pi/2}{(1-1/a^2)\,a^2\,(2i/a)}$$

$$= \frac{a\ln a + i\pi a/2}{(a^2-1)\,2i}$$

$$\therefore \oint_\Gamma f(z)\,dz = 2\pi i \times \left\{\frac{a\ln a + i\pi a/2}{2i(a^2-1)} + \frac{\pi}{4}\left(\frac{1}{1-a^2}\right)\right\} \quad \text{---} (3)$$

$$= \frac{-\pi a\ln a}{(1-a^2)} + \frac{i\pi^2}{2}\frac{1}{(a^2-1)}\,(a-1) = \frac{-\pi a\ln a}{(1-a^2)} + \frac{i\pi^2}{2(1+a)} \quad \text{---} (4)$$

Comparing real and imaginary parts of (1) and (4) we get

$$\int_0^\infty \frac{\operatorname{Log} x\,dx}{(x^2+1)(x^2a^2+1)} = \frac{-\pi a\ln a}{(1-a^2)}\ ; \quad \int_0^\infty \frac{dx}{(1+x^2)(1+x^2a^2)} = \frac{\pi}{2(1+a)}$$