Q11
§§7.4

To compute the integral

$$I = \int_0^\infty \frac{\sin^2 ax \cos^2 bx}{(\beta^2+x^2)}\, dx$$

using the method of contour integration.

This integral is straight forward if we write the numerator as follows

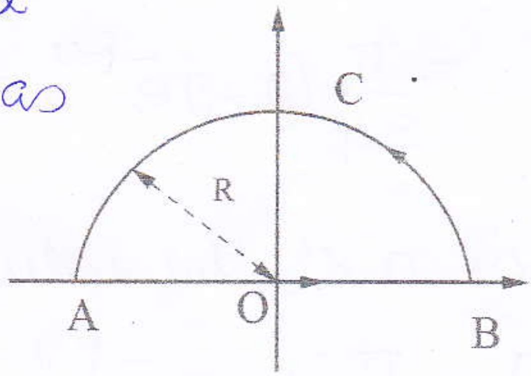

Fig. 1 Semi-circular contour

$$\sin^2 ax \cos^2 bx$$

$$= \frac{1}{2}(1-\cos 2ax)\frac{1}{2}(1+\cos 2bx)$$

$$= \frac{1}{4}(1-\cos 2ax+\cos 2bx - \cos 2ax \cos 2bx)$$

$$= \frac{1}{4}\left(1-\cos 2ax+\cos 2bx - \tfrac{1}{2}\cos(2(a+b)x) - \tfrac{1}{2}\cos(2(a-b)x)\right)$$

Each term in the above expression gives rise to an integral of ~~the~~ form

$$I' = \int_0^\infty \frac{\cos \lambda x}{(\beta^2+x^2)}\, dx$$

Evaluate I' by closing the contour in upper half plane

$$I' = \frac{1}{2}\,\mathrm{Re}\int_0^\infty \frac{e^{i\lambda x}}{(\beta^2+x^2)}\,dx = \frac{1}{2}\,\mathrm{Re}\oint_{AOBCA} \frac{e^{iz\lambda}}{(z^2+\beta^2)}\,dz$$

$$= 2\pi i \,\mathrm{Res} \left.\frac{e^{i\lambda z}}{(z^2+\beta^2)}\right|_{z=i\beta}$$

$$= \frac{\pi}{2\beta} e^{-\beta\lambda}$$

Using this result the required integral becomes $(a>b)$

$$I = \frac{\pi}{8\beta}\left(1 - e^{-2\beta a} + e^{-2\beta b} - \frac{1}{2}e^{-2\beta(a+b)} - \frac{1}{2}e^{-2\beta(a-b)}\right)$$

$$= \frac{\pi}{16\beta}\left(1 - e^{-2\beta a} + e^{-2\beta b} - \frac{1}{2}e^{-2\beta(a+b)} - \frac{1}{2}e^{-2\beta(a-b)}\right)$$

$$= \frac{\pi}{32\beta}\left(2 - 2e^{-\beta a} + 2e^{-\beta b} - e^{-2\beta(a+b)} - e^{-2\beta(a-b)}\right)$$

for $a<b$ the value of integral is seen to be $a>b$.

$$I = \frac{\pi}{32\beta}\left(2 - 2e^{-\beta a} + 2e^{-\beta b} - e^{-2\beta(a+b)} - e^{-2\beta(b-a)}\right)$$

$\therefore$ for all a, b positive

$$I = \frac{\pi}{32\beta}\left(2 - 2e^{-\beta a} + 2e^{-\beta b} - e^{-2\beta(a+b)} - e^{-2\beta|b-a|}\right)$$

$\boxed{\text{May 20, 2017}}$