§§7.3
Q5

Integrate

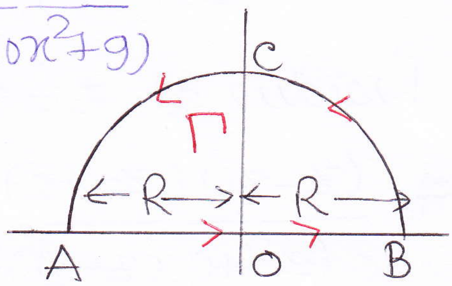
$$\int_{-\infty}^{\infty} \frac{(x^2+2)\,dx}{(x^4+10x^2+9)}$$

using the method of contour integration.

Solution:

We first write the integral as a limit of a finite interval $(-R, R)$ of real axis

$$\int_{-\infty}^{\infty} \frac{(x^2+2)\,dx}{(x^4+10x^2+9)} = \lim_{R\to\infty} \int_{-R}^{R} \frac{(x^2+2)\,dx}{(x^4+10x^2+9)}$$



$$= \int_{AOB} \frac{(z^2+2)\,dz}{(z^4+10z^2+9)}$$

A semicircular contour BCA can be added as it follows from the Darboux theorem that $\int_{BCA} \to 0$ as $R \to \infty$. Therefore

$$\int_{-\infty}^{\infty} \frac{(x^2+2)\,dx}{(x^4+10x^2+9)} = \lim \int_{AOB} + \lim \int_{BCA} \frac{(z^2+2)\,dz}{(z^4+10z^2+9)}$$

$$= \oint_{\Gamma} \frac{(z^2+2)\,dz}{(z^4+10z^2+9)}$$

The denominator factorizes as

$$z^4+10z^2+9 = (z^2+1)(z^2+9)$$
$$= (z-i)(z+i)(z-3i)(z+3i)$$

Poles enclosed by the contour are at $z = i, 3i$

We compute the residues at these poles

$$\operatorname{Res} \left.\frac{z^2+2}{(z^4+10z+9)}\right|_{z=i}$$

$$= \operatorname{Res} \left.\frac{z^2+2}{(z+i)(z-i)(z^2+9)}\right|_{z=i}$$ ← [z = i is a simple pole]

$$= \lim_{z\to i} \left.\frac{(z-i)\times(z^2+2)}{(z+i)(z-i)(z^2+9)}\right|_{z=i}$$

$$= \frac{(-1+2)}{2i(-1+9)} = \frac{1}{16i}$$

Residue at $z = 3i$

$$= \lim \left.\frac{(z-3i)(z^2+2)}{(z^2+1)(z-3i)(z+3i)}\right|_{z=3i}$$

$$= \frac{(-9+2)}{(-9+1)(6i)} = \frac{7}{8\times 6i}$$

∴ The value of integral

$= 2\pi i \times$ sum of residues

$$= (2\pi i)\left(\frac{1}{16i} + \frac{7}{48i}\right)$$

$$= \frac{2\pi i}{48i}(3+7) = \frac{20\pi i}{48i}$$

$$= \frac{5\pi}{12}$$

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The second integral $I_2 = -2\int_0^{\infty} \frac{dx}{(x^4+10x^2+9)}$

can be evaluated by closing the contour in the upper half plane

$$I_2 = -\int_{-\infty}^{\infty} \frac{dx}{(x^4+10x^2+9)} = -\oint_{C_R+} \frac{dz}{(z^4+10z^2+9)}$$

Only poles at $z = 3i, i$ contribute to the integral

$$\text{Residue}\left\{\frac{1}{z^4+10z^2+9}\right\}_{z=3i} = \left.\frac{1}{(z+3i)(z^2+1)}\right|_{z=3i}$$

$$= \frac{1}{(6i)(-8)} = \frac{i}{48}$$

$$\text{Residue}\left\{\frac{1}{z^4+10z^2+9}\right\}_{z=i} = \left.\frac{1}{(z^2+9)(z+i)}\right|_{z=i}$$

$$= \frac{1}{8(2i)} = \frac{-i}{16}$$

$$\therefore I_2 = -(2\pi i)\left(\frac{i}{48} - \frac{i}{16}\right) = 2\pi\left(\frac{1-3}{48}\right)$$

$$= -\pi/12$$

The final answer is $\left(\frac{1}{8}\ln 3 - \frac{\pi}{12}\right)$

$$= \frac{1}{24}(3\ln 3 - 2\pi) = \frac{1}{24}(\ln 27 - 2\pi)$$