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Dirac Delta Function Potential — Direct integration of the Schrödinger equation

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We integrate the Schrödinger equation to derive boundary condition on the derivative of the eigenfunction at $x = 0$. We rewrite the Schrödinger equation for the delta function potential

$$-\frac{\hbar^2}{2m}\frac{d^2u(x)}{dx^2} - g\delta(x)u(x) = Eu(x) \tag{1}$$

in the form

$$\frac{d^2u(x)}{dx^2} + \frac{2mg}{\hbar^2}\delta(x)u(x) = \frac{2mE}{\hbar^2}u(x). \tag{2}$$

We want to solve for the bound state energy, hence E is negative and we set $E = -|E|$ and rewrite the Schrödinger equation as

$$\frac{d^2u(x)}{dx^2} + \frac{2mg}{\hbar^2}u(x) - \alpha^2 u(x) = 0, \tag{3}$$

where $\alpha^2 = \frac{2m|E|}{\hbar^2}$.

Since $\delta(x)$ is zero for $x \neq 0$, the Schrödinger equation for $x < 0$ and $x > 0$, both, takes the form

$$\frac{d^2u(x)}{dx^2} - \alpha^2 u(x) = 0 \tag{4}$$

which has two independent solutions $e^{\alpha x}$ and $e^{-\alpha x}$ and we write the most general solution as

$$u(x) = \begin{cases} u_1(x) = A\exp(\alpha x) + B\exp(-\alpha x) & x < 0 \\ u_2(x) = C\exp(\alpha x) + D\exp(-\alpha x) & x > 0 \end{cases} \tag{5}$$

Taking the boundary condition $u(x) \to 0$ as $x \to \pm\infty$ we get $B = C = 0$ and the solution for the eigenfunction becomes

$$u(x) = \begin{cases} u_1(x) = A\exp(\alpha x) & x < 0 \\ u_2(x) = D\exp(-\alpha x) & x > 0 \end{cases} \tag{6}$$

Demanding that the wave function be continuous at $x = 0$ ($u_1(0) = u_2(0)$) gives $D = A$ and we get

$$u(x) = \begin{cases} u_1(x) = A\exp(\alpha x) & x < 0 \\ u_2(x) = A\exp(-\alpha x) & x > 0 \end{cases} \tag{7}$$

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We integrate the Schrödinger equation, Eq.(3), from $-\epsilon$ to ϵ and take the limit $\epsilon \to 0$.

$$\int_{-\epsilon}^{\epsilon} \frac{d^2u}{dx^2}\, dx + \frac{mg}{\hbar^2} \int_{-\epsilon}^{\epsilon} \delta(x)u(x)\, dx - \alpha^2 \int_{-\epsilon}^{\epsilon} u(x)\, dx \;=\; 0. \tag{8}$$

$$\text{or} \qquad \left.\frac{du}{dx}\right|_{-\epsilon}^{\epsilon} + \frac{2mg}{\hbar^2} u(0) - \alpha^2 \int_{-\epsilon}^{\epsilon} u(x)\, dx \;=\; 0. \tag{9}$$

The solution $u(x)$ is continuous at $x = 0$, hence in the limit $\epsilon \to 0$ the region of integration shrinks to zero and the last terms vanishes and we get the boundary condition on the first derivative at $x = 0$ as

$$\boxed{\left.\frac{du_2}{dx}\right|_{x=\epsilon} - \left.\frac{du_1}{dx}\right|_{x=-\epsilon} + \frac{2mg}{\hbar^2} u(0) = 0.} \tag{10}$$

Now using the explicit solution, Eq.(7), we get

$$u(0) = u_1(0) = u_2(0) = A \tag{11}$$

and

$$\left.\frac{du}{dx}\right|_{x=\epsilon} = \left.\frac{du_2}{dx}\right|_{x=\epsilon} = -A\alpha e^{-\alpha\epsilon}, \tag{12}$$

$$\left.\frac{du}{dx}\right|_{x=-\epsilon} = \left.\frac{du_1}{dx}\right|_{x=-\epsilon} = A\alpha e^{-\alpha\epsilon}. \tag{13}$$

The boundary condition, Eq.(10), in the limit $\epsilon \to 0$ becomes

$$-A\alpha e^{-\alpha\epsilon} - A\alpha e^{\alpha\epsilon} + \frac{2mg}{\hbar^2} A = 0 \tag{14}$$

$$\frac{2mg}{\hbar^2} = 2\alpha \Rightarrow \frac{m^2 g^2}{\hbar^4} = \alpha^2 = \frac{2m|E|}{\hbar^2}. \tag{15}$$

Thus we get the bound state energy as

$$|E| = \frac{mg^2}{2\hbar^2} \Longrightarrow E = -|E| = -\frac{mg^2}{2\hbar^2}. \tag{16}$$

The final form of the energy eigenfunction

$$u(x) = A \exp(-\alpha|x|), \tag{17}$$

after normalization

$$\int_{-\infty}^{\infty} |u(x)|^2\, dx = 1 \Rightarrow 2 \int_0^{\infty} |A|^2 \exp(-2\alpha x)\, dx = 1, \tag{18}$$

is given by

$$u(x) = \alpha^{-1/2} e^{-\alpha|x|}, \qquad \alpha = \frac{2mg}{\hbar^2}. \tag{19}$$

and the corresponding energy is given by Eq.(16).