Lessons in Quantum Quantum Field Theory

Some Mathematical Preparation

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Contents

§1 Lesson Overview

Syllabus

Defining operators $a_k, a_k^\dagger, N_k$;Commutation relations and properties of $a_k, a_k^\dagger, N_k$;Hilbert space of quantized Schrodinger field; Physical interpretation of states and field operators.

Objectives

1. To define operators $a, a^\dagger, N$ in terms of fields.
2. To derive commutation properties of $N, a, a^\dagger$
3. To define a complete orthonormal set.
4. To construct Hilbert space as linear span of the complete orthonormal set.
5. To give interpretation of field operator.

Prerequisites

Quantum Mechanics	Complete orthonormal set; Complete commuting set of hermitian operators; Harmonic oscillator.
Vector Spaces	Postulates of quantum mechanics; Simultaneous measurement; Specification of state.

§2 Recall and Discuss — Vector Spaces

§2.1 Complete orthonormal set

A set of vectors $\mathscr{O} = \{|\nu\rangle \big| \nu = 0, 1, 2, \ldots\}$is called a complete orthonormal set if

$$\langle\nu|\mu\rangle = \delta_{\mu\nu}, \tag{1}$$

and if there does not exist another orthonormal set $\mathscr{O}'$ such that $\mathscr{O} \subsetneq \mathscr{O}'$.

The importance of complete orthonormal set is that it can be taken as a basis and the Hilbert space is just the linear span of a complete othonormal set set.

§2.2 Complete commuting Set

A set of operators $\mathcal{S} = \{A_k\}$ is a commuting set if every pair of operators in $\mathcal{S}$ commute.

$$[A_j, A_k] = 0, \quad \forall j, k$$

The set $\mathcal{S}$ is a complete commuting set if an operator Y that commutes with all X_k, then Y is a function of X_k. The simultaneous eigenvectors of a complete set of commuting hermitian operators forms an orthonormal basis. Every vector in

Hilbert space can be written as a unique linear superposition of the simultaneous eigenvectors of complete commuting set of hermitian operators.

§3 Recall and Discuss — Quantum Mechanics

§3.1 Postulates of Quantum Mechanics

1. States of a Physical system are represented by vectors in a complex vector space with inner product (Hilbert Space).

2. Dynamical variables are represented by hermitian operators in the vector space.

3. Let A be dynamical variable and $\widehat{A}$ be corresponding operator having eigenvalues and eigenvectors $\alpha_n, |\alpha_n\rangle$.

   If state is represented by $|\alpha_k\rangle$ a measurement of A will give value α_k with probability 1. Conversely, if measurement of A gives α_k with probability 1, then the state is represented by the corresponding eigenvector $|\alpha_k\rangle$.

   If the state vector is an arbitrary vector $|\psi\rangle$, a measurement of A will give value α_k with probabilities $|\langle\psi|\alpha_k\rangle|^2$,

§3.2 Simultaneous measurement

A set of dynamical variables $A_1, A_2, \ldots$ can be measured simultaneously if they commute pairwise, *i.e.* $[A_j, A_k] = 0$ for all pairs A_j, A_k.

§3.3 State specification

Physically, a complete commuting set of observables can be measured simultaneously. The probability of all possible outcomes of such measurement is the maximum information that can be obtained about the state of the system from experiments. The state vector can be identified with the set of all such probability amplitudes.

§3.4 Harmonic Oscillator Operators $a, a^\dagger, N$.

1. The harmonic oscillator creation and annihilation operators obey the commutation rules
$$\left[a, a^\dagger\right] = 1. \tag{2}$$

2. The number operator defined as $N = a^\dagger a$ obeys the commutation relations
$$\left[a, N\right] = -a \qquad \left[a^\dagger, N\right] = a^\dagger. \tag{3}$$

3. The operator a and $a^\dagger$, decrease and increase eigenvalue of N by one.

4. The operator N is a positive definite operator and has eigenvalues $\nu = 0, 1, 2, ...$

5. The eigenvectors of N form a complete orthonormal set. The linear span of this set coincides with the Hilbert space.

§4 Some mathematics of Quantizated Schrödinger field

§4.1 Define Operators $a, a^\dagger, N_k$

Let $u_n(x)$ denote a complete set of orthonormal functions, assumed to be labeled by a discrete index n. This set can be chosen to be any set that may suit a given problem. The assumption that the eigenfunctions form an orthonormal set means

$$\int u_m^*(x) u_n(x)\, dx = \delta_{mn}$$

Completeness means that we can expand

$$\psi(x,t) = \sum_n a_n(t) u_n(x). \tag{4}$$

Since $\psi(x,t)$ has become operator in the second quantized theory, the expansion will be in terms of operators $a_n(t)$. Similarly we will have

$$\psi^\dagger(x,t) = \sum_n a_n^\dagger(t) u_n^*(x). \tag{5}$$

Using the orthogonality property of the eigenfunctions $u_n(x)$ we can invert the above relations and write

$$a_n(t) = \int u_n^*(x)\psi(x,t)\, dx, \qquad a_n^\dagger(t) = \int u_n(x)\psi^\dagger(x,t)\, dx \qquad \text{Verify} \tag{6}$$

The above equations can be looked upon as a change in coordinates from ψ, ψ^* to $a_n, a_n^\dagger$. The new variables $a_n, a_n^\dagger$ have a direct physical interpretation.

§4.2 Commutation properties of $a, a^\dagger, N_k$

We now use ETCR and Eq.(6) to compute the commutation relations satisfied by the operators $a_k, a_k^\dagger$. Consider

$$\begin{aligned}
[a_n(t), a_m^\dagger(t)] &= \left[\int u_n^*(x)\psi(x,t)dx, \int u_m(y)\psi^\dagger(y,t)dy\right] && (7)\\
&= \int dx \int dy u_n^*(x) u_m(y) \left[\psi(x,t), \psi^\dagger(y,t)\right] && (8)\\
&= \int dx \int dy u_n^*(x) u_m(y) \delta(x-y) && (9)\\
&= \int u_n^*(x) u_m(x)\, dx && (10)\\
&= \delta_{mn} && (11)
\end{aligned}$$

Thus we have

$$\boxed{\left[a_n(t), a_m^\dagger(t)\right] = \delta_{nm}} \tag{12}$$

In a similar fashion use of other commutation relations gives

$$\boxed{[a_m(t), a_n(t)] = 0, \qquad [a_m^\dagger(t), a_n^\dagger(t)] = 0, \qquad \text{for all } m, n} \tag{13}$$

The above commutation relations should remind you of the commutation rules

$$[a,a] = 0 \qquad [a, a^\dagger] = 1, \qquad [a^\dagger, a^\dagger] = 0. \tag{14}$$

You have encountered similar commutation relations for a harmonic oscillator; except that now there is one oscillator for each m.

> *Thus our system, the second quantized Schrödinger theory, looks like a collection of infinite number of harmonic oscillators.*

§4.3 Summary of Properties of $a, a^\dagger, N_k$

Define operators N_k, called number operators, by

$$N_k = a_k^\dagger a_k. \tag{15}$$

The following results can be derived easily by following the steps you have learned for $a, a^\dagger$ in harmonic oscillator.

1. The operators N_k are hermitian, *i.e.* $N_k^\dagger = N_k$.
2. $[N_k, N_k] = 0$.
3. $[N_k, a_k] = -a_k$ and $[N_k, a_k^\dagger] = a_k^\dagger$. Verify!
4. Each operator N_k has eigenvalues 0, 1, 2,. . . .

§5 Constructing the Hilbert Space

§5.1 Choice of an o.n. basis

The fact that the operators N_k commute pairwise implies that they will have simultaneous eigenvectors which will form a basis. We denote this set of eigenvectors by $\mathscr{B} = \Big\{|\nu_1, \nu_2, \ldots\rangle \Big| \nu_1, \nu_2, ... = 0, 1, 2, 3, \ldots\Big\}$ will form a complete orthonormal set. A vector in this set corresponds to eigenvalues $\nu_1, \nu_2 \ldots$ for operators $N_1, N_2, \ldots$.

$$\begin{aligned} N_1|\nu_1, \nu_2, \ldots\rangle &= \nu_1|\nu_1, \nu_2, \ldots\rangle & (16)\\ N_2|\nu_1, \nu_2, \ldots\rangle &= \nu_2|\nu_1, \nu_2, \ldots\rangle & (17)\\ \ldots \quad \ldots & \quad \ldots \quad \ldots & \\ N_k|\nu_1, \nu_2, \ldots\rangle &= \nu_k|\nu_1, \nu_2, \ldots\rangle & (18)\\ \ldots \quad \ldots & \quad \ldots \quad \ldots & \end{aligned}$$

The state $|\nu_1, \nu_2, ...\rangle$ has the interpretation of ν_1 particles in state $u_1(x)$, ν_2 particles in state $u_2(x)$ and so on.

§5.2 The vacuum state

We assume that there exists a state $|0\rangle$, unique up to a phase, such that

$$a_|0\rangle = 0, \text{ for all } k = 1, 2, 3, ... \tag{19}$$

The eigenvectors of N_k can now be constructed by applying products of suitable powers of $a_k^\dagger, k = 1, 2, ..$ on the vacuum. Thus

$$\begin{aligned} |\nu_1, 0, 0, ...\rangle &= (a_1^\dagger)^{\nu_1}|0\rangle & (20)\\ |\nu_1, \nu_2, 0, 0, ...\rangle &= (a_1^\dagger)^{\nu_2}(a_2^\dagger)^{\nu_2}|0\rangle & (21)\\ |\nu_1, \nu_2, \ldots, \nu_k, \ldots\rangle &= \left(\prod_k \frac{a_k^{\nu_k}}{\sqrt{\nu_k!}}\right)|0\rangle. & (22) \end{aligned}$$

The completeness of the basis $\mathscr{B}$ is expressed by the following relation

$$\sum_{\nu_1, \nu_2, \ldots} |\nu_1, \nu_2, \ldots, \nu_k, \ldots\rangle\langle\nu_1', \nu_2', \ldots, \nu_k', \ldots| = \delta_{\nu_1\nu_1'}\delta_{\nu_2\nu_2'}\ldots\delta_{\nu_k\nu_k'}\ldots \tag{23}$$

§6 Physical Interpretation

§6.1 States of Quantized Schrodinger field

The simultaneous eigenvectors of the number operators constitute an orthonormal basis in the Hilbert space of the quantized Schrodinger field theory

Given this set of basis vectors, we can construct the Hilbert space by taking all possible linear combinations of the elements of the complete orthonormal set. A representation obtained by taking $\{\mathscr{B}\}$ as basis is called *number representation.*

An arbitrary state $|\Psi\rangle$ in the Hilbert space has the interpretation that the probability amplitude that there are $\nu_1, \nu_2, \ldots$ particles, respectively, in states $u_1, u_2, \ldots$is given by $\langle \nu_1, \nu_2, \ldots, \nu_k, \ldots |\Psi\rangle$.

§6.2 The field operator

Like the wave function in quantum mechanics, the field operator itself is not a measurable physical quantity. In quantum mechanics the interpretation of the wave function comes through the probability density and the probability current density. Corresponding operators and the number operator N, in the second quantized theory have the interpretation as explained below.

The operator $N = \sum_k N_k$ represents the number of particles. **In case of Schrodinger field the total number of particles is conserved.** It is easy to verify that the total number operator N expressed in terms of the Schrodinger field $\psi(x)$ takes the form

$$N = \int \psi^*(\mathbf{x})\psi(\mathbf{x})\, d\mathbf{x} \tag{24}$$

Therefore $|\psi(x)|^2$ has the interpretation of being number density,or the number of particles per unit volume.

§7 EndNotes

§7.1 Points of Discussion

§7.2 More about Harmonic Oscillator Connection

We will now recall and rewrite equations from harmonic oscillator which show that each N_k is harmonic oscillator Hamiltonian. For this purpose we introduce operators q_k, p_k by

$$q_k = \frac{1}{\sqrt{2}}(a_k^\dagger + a_k), \quad p_k = \frac{i}{\sqrt{2}}(a_k^\dagger - a_k) \tag{25}$$

These operators obey commutation relations

$$\big[q_k, q_\ell\big] = \big[p_k, p_\ell\big] = 0, \qquad \big[q_k, p_\ell\big] = i\delta_{k\ell}. \tag{26}$$

The number operators N_k written in terms of q_k, p_ktake the form

$$N_k = \frac{1}{2}\big(q_k^2 + p_k^2\big) - \frac{1}{2} \tag{27}$$

Thus each N_k, apart from a constant, is like harmonic oscillator Hamiltonian.

§7.3 Points to Remember

- ✍ The simultaneous eigenvectors of N_k are chosen as an orthonormal basis to get the Hilbert space .

- ✍ The simultaneous eigenvectors correspond $|\nu_1, \nu_2, ..\rangle$ to the state in which there are $\nu_1, \nu_2...$ particles in 'levels' $u_1, u_2, ...$, respectively .

- ✍ The physical interpretation of the vectors in the Hilbert space in the number representation depends on the choice of basis functions $\{u_n(x)\}$ for expansion of field operator.

- ✍ Recall the equation of continuity in quantum mechanics

$$\frac{d\rho}{dt} + \nabla \cdot \mathbf{J} = 0 \tag{28}$$

The 'probability density' and the 'probability current' from quantum mechanics now have the following particle interpretation:
$\langle\alpha\rangle \rho = \psi^\dagger(\mathbf{x})\psi(\mathbf{x})$ as number of particles per unit volume
$\langle\beta\rangle$ $J(\mathbf{x}, t) = \frac{i\hbar}{2m}(\psi^\dagger(\nabla\psi) - (\nabla\psi^\dagger)\psi$ as flux.